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CHAIN THEOREMS IN GEOMETRY.*

CHAINS OF THE COX-GRACE TYPE.

BY

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[Received 5 April, 1945.]

I Introduction.

1.0. A Chain theorem in geometry is a sequence of theorems, each affirming the existence of a geometrical construct derived from a number of primary elements. The construct at any stage is used to obtain that at the next stage, the resulting configuration growing with the number of initial primary elements. An important type of these chains is the Cox-chain† and the chain associated with the names of Miquel, Clifford, Cantor and Grace.

1.1. The Miquel-Clifford chain associates with a set of n lines, a point called their Miquel point or a circle-their Clifford circle according as n is even or odd. Out of these n lines omitting one at a time we get n subsets of $n-1$ lines each, such that (as n is even or odd) their Clifford circles are concurrent at the Miquel point of the n lines or their Miquel points lie on a circle—the Clifford circle of the n lines.

1.11. The Miquel point of two lines is their point of intersection and the Clifford circle of three lines is the circumcircle of the triangle formed by them. It is known that given 4 lines the Miquel circles of the sets of 3 of these are concurrent.‡ Miquel[2] proved—the next

* I wish to express my thanks to Dr. A. Narasinga Rao for his constant help and guidance.

† A reference to Cox-chain is given by Baker,[8] Morley and Morley *Inversive Geometry*, p. 271 and by Richmond.[18]

‡ This was first given by Wallace[7] [1799] and can be a cor. of Lamberts' theorem : 'The circumcircle of a triangle circumscribed to a parabola passes thro' the focus', [1761].

link in the chain—that the points of concurrence of the 5 quadrilaterals derived from 5 lines are concyclic and finally Clifford[3] and Kantor[4] showed that this alternation of concurrent circles and concyclic points continues indefinitely.

1.12. Their proof depends on a generalisation of Lambert's theorem to m-parabolas (curves of class $m+1$ touching the line at infinity m times) such a curve is determined uniquely by $2m+2$ tangents. Its focus is the Miquel point of the $2m+2$ lines. Touching $2m+1$ lines, there is a tangential pencil of m-parabolas and the tangents to these from the circular points at infinity form corresponding pencils of two homographic pencils. Hence their intersections—the foci of the various m-parabolas—lie on a circle, the Clifford circle of the $2m+1$ lines. On the other hand with $2m+3$ lines, we have a Clifford circle, passing among others, through the foci of the degenerate $m+1$ parabolas consisting of the point at infinity on any one of these lines and the m-parabola touching the $2m+2$ other lines.

1.2. G. H. Grace[6] appears to have been the first to point out that the above configuration gains in symmetry by inversion, becoming one of 2^{n-1} points and 2^{n-1} circles on a sphere—or its topological equivalent, the inversive plane—such that n points lie on each circle and n circles pass through each point.

1.3. The important features of this paper are: (*a*) Proving that any chain of theorems whose incidences follow scheme A will follow indefinitely if we can prove that the theorems of the chain are true for four consecutive stages; and that the complete configuration—the truncated Grace configuration—is the resultant of a number of overlapping (complete) Grace configuration superposed—or alternatively is a Grace configuration from which a certain portion is cut off. (*b*) Proving a similar result for chains of the type of the projective chain of

theorems due to Cox; and deriving as particular cases, the Grace chain and the Dual Grace chain. (*c*) Deriving from (*a*) a number of known chains; the point-O-theorem chain, the nine-points circle chain, the orthocentre chain, the oriented line chain, the Coolidge's oriented circle chain, Narasinga Rao's bisector k-circle chain, and orthogonal k-circle chain, Richmond's line congruence chain, etc. (*d*) Finally drawing attention to certain new chains, the pedal circle chain, the rectangular hyperbola chain, the radical axis chain, the oriented k-circle chain, the T-turbine chain, etc.

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II NOTATION AND INCIDENCE SCHEME.

2.0. A simple and suggestive notation bringing out the incidences and structure is of the utmost help in the study of any configuration. The notation described in §§ 2.3 and 2.4, introduced tentatively by Grace and systematized by Narasinga Rao is found to be very convenient.

2.1. A consistent use of a scheme of notation has been made by Neville for the study of the Grace-chain. He denotes points by ordered sets of $n-1$ symbols each of which is α or β and circle by sets of n symbols each of which is A or B. A circle and a point are incident if, apart from the interchange of the Greek and Roman letters, their symbols differ in not more than one place. Thus the circles incident with the point $(\alpha\beta\alpha)$ are (ABA), (BBA), (AAA) and (ABB).

2.2. We can pass from the notation of § 2.4 to that of Neville by making a circle or a point $(i, j, \ldots, l)$ in the Grace configuration of § 2.6 correspond to an ordered set of $n-1$ letters each of which is α (A), with however the i th . . . and the l th places changed to β (B). Similarly to pass from Neville's to ours the point (circle) denoted by the ordered set of α and β $(A$ and $B)$ is made to correspond to $(i, j, \ldots, l)$, where the β's $(B$'s) occupy the i th, ... l th places.

2.3. We start with a number n of 'primary' elements of a certain set S, besides possibly other elements fixed in number, and derive from them in succession, as n increases, two types of derived elements called 'points' and 'circles' connected by a binary relation 'incidence'. The primary elements are denoted by letters 1, 2, 3,... and the element derived from these $i, j, k,\ldots, l$ is denoted by $(i\,j\,k\ldots\,l)$. If we associate a 'point' or a 'circle' even without any primary element to start with, i.e. with zero primary elements, we denote it by the bracket symbol (). The notations and incidences characteristic of the type of chains discussed here are given by the following :

2.4. INCIDENCE SCHEME A : If at least p primary elements are needed to derive a circle or a point of the chain, then

2.41. *Every bracket symbol consisting of p or more letters (all distinct) denotes a circle or a point, according as the number of letters in the symbol is odd or even.*

2.42. *The order of the letters in a bracket symbol is immaterial.*

2.43. *A point and a circle are incident if and only if their bracket symbols are the same except for an additional letter in one of them.*

2.44. *The bracket symbol $(i\,j\ldots l)$ and $(i\,j\,\ldots\,l\,k\,k)$ are identical, i.e. the repetition of a letter which occurs is equal to its omission. Thus, (i) is incident with $(i\,1)$, $(i\,2)$, ..., $(i n)$ including $(i\,i) = ()$.*

2.5. THE TRUNCATED GRACE CONFIGURATION OF ORDER n AND DEFICIENCY p:—Thus starting with n elements in the above case, the final configuration will consist of (*i*) ${}_nC_p+{}_nC_{p+2}+{}_nC_{p+4}+\ldots$ circles (points) each of which—excepting ${}_nC_p$ which are incident with only $n-p$ points (circles)—is incident with n points (circles) and (*ii*) ${}_nC_{p+1}+{}_nC_{p+3}+{}_nC_{p+4}+\ldots$ points (circles) each of which is incident with n circles (points).

2.6. GRACE CONFIGURATION consisting of 2^{n-1} circles and 2^{n-1} points, each circle passing through n points and each point lying on n circles is got when $p = 0$. The configuration is completely symmetric in that all the points (circles) have the same status, whereas, in the truncated Grace configuration the points (circles) fall into sets of equivalent elements ${}_nC_r$ in number $r = p, p+1, \ldots$

III COX CHAIN AND COROLLORIES.

3. In §§ 3.1 to 3.8 bracket symbols denote planes, and points on a quadric in a projective S_3 and not circles on a plane.

3.1. LEMMA 1. *Let* (1), (2), (3), (4) *be four arbitrary planes passing through a point* (). *On the line of intersection of any pair of these, say* (i), (j) *take an arbitrary point* (ij). *Let* (ijk) *be the plane through the points* (ij), (jk), (ki). *Then the four planes* (123), (124), (134) *and* (234) *concur at a point* (1234).

PROOF. It is well known that all quadrics through seven of the eight intersections of three linearly independant quadrics, pass also through the eighth. Consider now the points (), (12), (13), (14), (23), (24), (34) and P the point of intersection of the planes say (123), (124) and (134). The plane pairs (2), (134); (3), (124); (4), (123) constitute three linearly independant* quadrics, through these eight points. Hence the plane pair (1), (234) passing through (1), (12), .., (134) must pass through P, i.e. the plane† (234) passes through the point of concurrence of the planes (123), (124) and (134).

3.2. Incidentally, we obtain that if the seven points (ij) and () lie on a quadric Ω, (1234) must also lie on Ω.

3.3. LEMMA 2 is the dual of Lemma 1.

3.4. FUNDAMENTAL THEOREM A. *If with n given elements* 1, 2, .., *n of a given type T, in S_3, it is possible to*

* Otherwise all their common points will lie on a line or a plane.

† Since evidently (1) does not pass through (1234).

associate when $n = N, N+1, N+2,$ *or* $N+3$ *respectively a point or a plane* $(12\ldots n)$ *according as n is odd or even, such that the* n' *points (planes)* $(i_1, i_2\ldots i_{r-1}\, i_{r+1}\ldots i_{n'})$ *are incident with the plane (point)* $(i_1\, i_2\ldots i_{n'})$ *where* $n' = N+1, N+2$ *or* $N+3$, *and the i's are the letters* 1, 2,..., *n, then can such an association of point or circle with n elements be made for all* $n \geq N$ *depending on n being odd or even and satisfying similar incidence relations.*

For, suppose first that $(12\ldots N)$ is a point. Then given $N+4$ elements $1, 2,\ldots N+4$, consider the planes $(12\ldots N\,N+i)$ $i = 1,\ldots 4$, passing through the point $(12\ldots N)$. From the data, $(12\ldots N\,N+i\;N+j)$ are points on the intersection of these planes in pairs and are connected in triads by the planes $(12\ldots N\;N+1\;N+2\;N+3)$ $(12\ldots N\;N+1\;N+2\;N+4)$ $(12\ldots N\;N+1\;N+3\;N+4)$ and $(12\ldots N\;N+2\;N+3\;N+4)$. Hence by Lemma 1 these are concurrent; whence by symmetry [by starting with the points $(23\ldots N+1)$, etc. instead of $(12\ldots N)$] all the $N+4$ planes $(12\ldots i-1\;i+1\ldots N+4)$ are concurrent at a point $(12\ldots N+4)$.

Similarly when $(12\ldots N)$ is a plane, we get from Lemma 2 that the points $(12\ldots i-1\;i+1\ldots N+4)$ lie on a plane $(12\ldots N+4)$.

Thus, if the association of incident points and planes as in §3.4 can be made for four consecutive links, the association can be continued for the next stage and hence by induction, indefinitely.

3.5. From §3.3, we get that if the points (planes) for four consecutive stages are incident with a quadric, so are the points (planes) at all further stages.

Since Lemma 1 gives four consecutive links of such a chain as in §3.4, it follows that these incidences continue indefinitely. This gives us the

3.6. Cox chain. *Let* (1), (2), ... *be planes through a point* () *in a projective* S_3. *On the intersection of every pair of these planes* (i), (j) *take an arbitrary point; let* (ijk) *be the plane through the points* (ij), (jk) *and* (ki). *Then*

*With four planes (1),..., (4) we have four planes (123), (124), (134) and (234). These will pass through a common point (1234).**

With five planes (1),..., (5), the five points like (1234) lie on a plane (12345) and so on.

With n planes (1),..., (n) through (), we arrive finally at (12..n) which is a point or a plane according as n is even or odd.

3.6′. From §3.3, we get, if all the points (*ij*) lie on a quadirc Ω through (), then shall all the subsequent points also lie on Ω.

3.7. Dual Cox chain: *Let (1), (2),... be points on a plane () in a projective S_3. Let (ij) be a plane through the points (i), (j). Let (ij), (jk), (ki) intersect at (ijk). Then with n points (1), (2), .. on (), we have n planes (points) (12.. i−1 i+1..n) incident with a point (plane) (12..n) according as n is odd or even.*

3.7.′ As before, if † we take all the points (*i*) and all the ${}_nC_3$ points (*ijk*) to lie on a quadric Ω, then all the subsequent points also lie on Ω.

3.8. It may be noted that the configurations obtained for the Cox chain and its dual are identically the same and consist of 2^{n-1} points and 2^{n-1} planes, each point lying on *n* planes and each plane passing through *n* points.

Taking S_3 to be the image of the tetracyclic plane and Ω to be the assemblage of the quadratic system‡ of

* For a neat analylical proof vide Baker[8]. The following method is indicated by Richmond[10]: Since eight points always lie on a quadric, the four-plane theorem is equivalent (making use of the usual correspondence between plane and points on a quadric Ω in S_3; and circles and points in a tetracyclic plane) to the four-circle theorem in Grace chain or the four-line theorem in the Clifford chain. But this is an immediate corollory of Lambert's theorem or of the pedal line theorem.

† That it is possible to satisfy these conditions is proved in §3.16.

‡ For examples of cases where the quadric represents assemblages other than circles of zero-radius see §7.

circles—the circles of zero radius, viz. points, we have from 3.6, 3.6′ and 3.7′ the following theorems.

3.9. GENERALISED GRACE CHAIN: *Let* (1), (2),.. *be a series of circles orthogonal to a given circle* (). *Let* $(i\,j)$ *be any circle orthogonal to* (i) *and* (j). *Let* $(i\,j\,k)$ *be the circle orthogonal to* (ij), (jk) *and* (ki). *Then with four circles* (1), (2), (3), (4), *the four circles* (123),...*will have a common orthogonal circle* (1234). *With five circles* (1),..., (5) *orthogonal to* (), *the five circles* (1234), *etc., have a common orthogonal circle* (12345). *and so on.*

3.10. THE GRACE CHAIN: *Let* (1), (2),..*be a series of circles concurrent at* (). *Let the circles* (i), (j) *intersect again at* (ij) *and let* (ijk) *be the circle through* (ij) (jk) *and* (ki).

Then with n circles (1)..(n), *we obtain finally* $(12\ldots n)$ *which is a circle or a point according n is odd or even, incident with the n points (circles)* $(12\ldots\, i{-}1\ i{+}1\ldots n)$.

3.11. THE DUAL GRACE CHAIN: *Let* (1), (2),..*be a series of points on a circle* (). *Let* (ij) *be a circle through* (i) *and* (j) *such that the circles* (ij), (jk), (ki) *are concurrent at* (ijk). *Then, with n points* (1),..., (n) *we obtain finally* $(12\ldots n)$ *which is a point or a circle according as n is odd or even, incident with the n circles (points)* $(12\ i{-}1\ i{+}1\ldots n)$.

3.12. THE GRACE CONFIGURATION (COMPLETE): Both the chains 3.10, 3.11 give rise to the same configuration—and are thus equivalent—consisting of 2^{n-1} points represented by all bracket symbols with an even (odd) number of letters and 2^{n-1} circles represented by all bracket symbols with an odd (even) number of letters, such that each point lies on n circles and each circle passes through n points. We call this the Grace configuration of order n*.[6] All the points in the configuration are equivalent in the sense that it can be generated equally

* Named after Grace who was the first to consider the configuration. Neville[10] has also drawn attention to the same independently.

from all the circles through any of the points. Thus to generate the configuration from all the circles through the point (1234), we add* (1234) to all the bracket symbols (), (i), (ij), (ijk), etc. in view of the convention that a letter repeated twice may be dropped.

Now, Grace's chain enables us to generate the entire configuration from the n circles through any of the n points. But, as Neville[10] has remarked, the n points on one of the circles do not constitute sufficient data to fix the configuration. The extra data needed to fix this is given by 3.11 in the dual Grace chain—the complement of the Grace chain. This will be brought more in evidence in the proof of the fundamental Theorem A (4.1), where these two chains together cover two complementary cases.

3.13. The dual Grace chain may, however, be considered to be an extension of the Grace chain in the sense that we can derive from it, as a particular case, Grace's† point-O-chain—itself an extention of the Clifford chain—which is inversively the same as the Grace chain.

Thus, let (1), (2) ... be a series of points on a circle () and let 1, 2,.. be arbitrary nonparallel lines through these. Let (ij) be the circle through the points (i), (j) and the intersection of the lines i, j. Then, by the point-O-theorem, every triad of circles (ij), (jk), (ki) is concurrent at (ijk). Thus the conditions of 3.11 are satisfied and we get

3.14. GRACE'S POINT-O-CHAIN:—*Let* 1, 2,.. *be a series of arbitrary nonparallel lines. Mark off concyclic points on these. Then,*

* *Vide* Narasinga Rao(19).

† Not only does Clifford's chain come out of this as a particular case when the marking circle becomes infinite in radius, but this chain is capable of extension to higher spaces, whereas, the Clifford chain is not.

2

With two lines 1, 2 *we have a circle* (12) *through* (1), (2) *and the intersection of* 1, 2 ; *and finally with* n *lines* 1, . ., n *we have* n *points (circles)* (12 . . $i-1$ $i+1$ . . $n+1$) *which lie on a circle (pass through a point)* (12 . . n) *according as* n *is even or odd.*

It may be noted that the only purpose served by the lines 1, 2,. . is to ensure that the circles (ij), (jk), (ki) are concurrent; and that accordingly the configuration is not altered by rotating the lines i about the points (i) through the same angle θ.

3·15. CLIFFORD CHAIN : *Let* (1), (2) . . *be a series of lines. Let* (ij) *be the intersection of the lines* (i), (j) *and let* (ijk) *be the circle through the points* (ij), (jk), (ki). *Then, with* n *lines* (1), (2). . (n), *according as* n *is even or odd, we get* n *circles (points)* (12. . $i-1$ $i+1$ . . n) *having common incident point or circle* (12 . . n).

This is a particular case of 3.4 when the circle on which the marked points lie, becomes infinite in radius.

3·16. In 3.7 we imposed 2 ${}_nC_2+{}_nC_3$ conditions on ${}_nC_2$ circles (each passing through a pair of points and ${}_nC_3$ triads of circles being concurrent). In Grace's point-O-chain, we have seen that it *is* possible to find sets of circles satisfying all these conditions. We can, in fact, choose all the circles (1i) through one of the points (1) arbitrarily, when all the circles (ij) will be determined by the conditions that (ki), (kj), (ij) should be concurrent.

For, suppose that (1i), (1j), (1k) are three circles through (1) and through (i), (j), (k) respectively. Then the circle (ij) is determined as passing through (i), (j) and the intersection (1ij) other than (1) of the circles (1i) and (1j). Similarly, the circles (jk), (ki) are determined as passing through (j), (k), (1jk) and (k), (i), (1ki) respectively. Then it follows by applying the inverted point-O-theorem to the circles (1i), (1j), (1k) concurrent

at (1) and having (i), (j), (k) as marked points thereon, that the circles (ij), (jk), (ki) are concurrent at (ijk).*

IV. Chains of the Grace Type.

4.0. The Fundamental Theorem A: *Let four consecutive links of a chain following the incidence scheme A be true. Then all the subsequent links in the chain are true.*

4.1. Thus, let it be given that:

(4.11) given any p primary elements $1, 2, .. p$ it is possible to derive by means of some specified procedure a unique circle (point) $(12 .. p)$;

(4.12) given any $p+1$ primary elements $1, 2, .. p+1$, the $p+1$ circles (points) $(12 ... r-1\ r+1 ... p+1)$ are incident with a point (circle) $(12 ... p+1)$;

(4.13) given any $p+2$ primary elements $1, 2, ... p+2$, the $p+2$ points (circles) $(12 ... r-1\ r+1 ... p+2)$ are incident with a circle (point) $(12 .. p+2)$ and

(4.14) given any $p+3$ primary elements $1, 2, ... p+3$, the $p+3$ circles (points) $(12 ... r-1\ r+1 .. p+3)$ are incident with a point (circle) $(12 ... p+3)$.

Then we shall show that given any $p+4$ 'primary' elements, $1, 2, .. p+4$, the $p+4$ points (circles) $(12 ... r-1\ r+1 ... p+4)$ are incident with a circle (point) $(12 .. p+4)$. And so on, indefinitely.

For, consider the points (circles) $(12 ... pi)$, $i \neq j \neq k = p+1, ..$ which are by 4.12 incident with $(12 ... p)$. Incident with pairs of these, by 4.13 we have $(12 ... pij)$ and by

* Similarly, let (i) be a system of points on a circle () and (ij) circles through pairs of these, such that (ij), (jk), (ki) are concurrent. Let a line 1 be drawn through (1) meeting the circle $(1i)$ at the point $1i$. Then it will follow that the joins of (i) to $1i$ and (j) to $1j$ meet at a point ij on the circles ij, i.e. given any such set of points and circles as we start with the dual Grace chain, it is possible to introduce straight lines through these points, such that the whole Grace configuration is obtained as the point-*O*-theorem configuration for these lines and these marked points. It is curious how each of these—the point-*O*-chain, the Clifford chain, the Grace chain and the dual Grace chain can be derived each from the other.

4.14, $(12\ldots pij)$, $(12\ldots pjk)$, $(12\ldots pki)$ are incident with the point (circle) $(12\ldots pijk)$. Hence by the four-elements theorem in the dual Grace chain (Grace chain), we get that the points (circles) $(12\ldots p.p+2\,p+3\,p+4)$, $(12pp+1.\,p+3\,p+4)$, $(12\ldots pp+1\,p+2.p+4)$ and $(12\ldots pp+1\ p+2\ p+3.)$ lie on a circle (pass through a point), which we shall temporarily call $(12\ldots p+4)$. Now, we could have started with $(23\ldots p+1)$ instead of $(12\ldots p)$ and obtained similarly that $(.23\ldots p+4$, $(12\ldots p+3.)$ $(12\ldots p+2.\ p+4)$ and $(12\ldots p+1.\ p+3\ p+4)$ are incident with a circle (point), which, from the above can only be $(12\ldots p+4)$. Similarly, by starting with elements in which, say, four consecutive letters are omitted say $(34\ldots p+2)$, $(45\ldots p+3)$, $(56\ldots p+4)$ $(1, 68\ldots p+4)$ and so on, we get that all the elements $(12\ldots i-1\ i+1\ldots p+4)$ are incident with $(12\ldots p+4)$. This proves the next link in the chain. Incidentally we see that the entire configuration can be obtained by the superposition of $p+4$ overlapping configurations of order 4.

Now, with four consecutive links $p, p+1$, $p+2$, $p+3$, we are able to prove the link $p+4$. Similarly from the links $p+1, \ldots p+4$, we get that the link $p+5$ is true; and so on indefinitely (i.e.) we have been able to associate with any symbol with p or more letters a circle (point) such that this, with bracket symbols of lower order satisfies incidence Scheme A. This proves the fundamental theorem. Incidentally, we obtain the additional

4.2. THEOREM. *The entire truncated Grace configuration of 2.5 corresponding to the chain of the fundamental Theorem 4.1, when we start with $n\ (=p+r)$ elements can be obtained by the superposition of ${}_nC_p$ Grace configuration of order r.*

4.3. Evidently this truncated configuration can also be obtained by removing from a (complete) Grace configuration of order n, ${}_nC_{p-1}$ Grace configurations of order $p-1$. For to complete the Grace configuration, we have to insert the elements $()$, (i), (ij), $(ijk)\ldots(i_1\,i_2,\ldots,\ i_{p-1})$

which in the truncated configuration do not represent any point or circle. All the elements in this configuration do not however hold the same footing, there being one, only one, central element $(12\,.\,.\,n)$, the others being distributed about it in equidistant and equivalent set of ${}_nC_k$. Thus, it is easily verified that, whereas the group of the Grace configuration[19] is the permutation group on n letters multiplied by the abelian group of order 2^n (2^{n-1}) according as circles and points may or may not be interchanged, the group of the truncated configuration is simply the permutation group on n letters.

4.4. When $p = 0$, we obtain the Grace chain and the dual Grace chains themselves. Thus the fundamental theorem is true for non-negative integral values of p, including zero.

V. Chains Derived from Theorem A.

5.1. The pedal circle chain: *Let P, 1, 2, . . be a series of coplanar points. Associate with three points 1, 2, 3 the pedal circle (123) of P w.r.t. the triangle formed by them. With four points* 1, 2, 3, 4 the four pedal circles like (123) will pass through a point (1234) and so on. With n points we get finally $(12\,.\,.\,n)$ which is a point or a circle according as n is even or odd.*

For, suppose with $p\,(= 0)$ points to start with, we associate () with the point P itself.

With $p = 1$ point i to start with, we can associate the circle (i) on $P\,i$ as diameter.

With two points i, j the circles (i), (j) intersect again at (ij) the projection of P on ij.

With three points i, j, k the points (ij) (jk), (ki) lie on a circle (ijk)—the pedal circle of P w.r.t. the triangle formed by i, j, k.

* This result has been noted (without proof) by V. Ramaswami Iyer, *The Mathematics Student*, 3 (1935).

Thus we have four consecutive links of a chain following incidence scheme A and hence by the fundamental Theorem A* 4.1, the chain is true for all stages.

The pedal circle of the circum-centre of a triangle is its nine-points circle. Hence, if all the points i lie on a circle with centre $P = ()$, the pedal circles (ijk) become nine-point circles of the triangles (i, j, k). Thus we have the following

5·2. NINE-POINTS CIRCLE CHAIN: *Let* 1, 2, . . *be a series of points on a circle C. Then, with three points* 1, 2, 3 *we have a nine-point circle* (123). *With four points* 1, 2, 3, 4 *the four nine-points circles like* (123) *are concurrent. Finally, with n points we have* $(12\,.\,.\,n)$ *which is a point (circle) incident with the circles (points)* $(12\,.\,.\,i-1\;i+1\,.\,.\,n)$ *according as n is even or odd.*

5·3. THE ORTHOCENTRE CHAIN: *Let* (1), . . . *be a series of points on a circle* (). *With three points* 1, 2, 3 *we have an orthocentre* (123). *With four points* 1, . . ., 4 *on* (), *the four orthocentres* (123), . . ., *lie on a circle* (1234) *and so on.*

For, with p $(=0)$ elements to start with, we can associate (). With one element (i), we can associate (i). With two elements (i), (j) we can associate the circle (ij) which is the reflection of the circle () in the line joining (i) to (j). This evidently passes through (i) and (j), and is seen† to be the locus of orthocentres of triangles two of whose vertices are (i), (j) and the third is on (). Hence, it follows that the circles (ij), (jk), (ki) concur at (ijk) the orthocentre of the triangle formed by the joins

* Appeal to the fundamental theorem is not essential. We are merely stating Grace chain for the circles (1), (2) . . through (). This is exactly the Grace chain with a change in view-point. We interpret the Grace chain as a property of one of the points of the configuration and the diameterically opposite points on the circles through it.

† Since, the reflections of the orthocentre of a triangle in the sides lie on the circumcircle.

(i), (j), (k). Thus we have four consecutive links of chain 5.3 and so the chain is seen to be true indefinitely. Here, as before, we are merely giving the dual Grace chain in a special case.

Since the nine-points centre is the midpoint of the join of the circum-and ortho-centres, we obtain by applying a similarity transformation centre at the circum-centre and ratio 1/2, the

5.4. NINE-POINTS CENTRE CHAIN: *Let* 1, 2, 3... *be a series of points on a circle* ().

With three points, 1, 2, 3 *we have a nine-points centre* (123).

With four points 1, 2, 3, 4 *the four nine-point centres like* (123) *lie on a circle* (1234) *and so on**.

If in the above, the ratio of the transformation be some constant $(k-1)/k$ *instead of* $\frac{1}{2}$†, *we obtain the*

5. 5. EULER CENTRE CHAIN‡: *Let* 1, 2,...*be a series of points on a circle*().

With three points i, j, k *we have an Euler centre for any fixed ratio say* (k) *which we can call* (ijk).

With four points i, j, k, l *the four Euler centres* (ijk),... *lie on a circle* $(ijkl)$ *and so on.*

THE CENTROIDAL CIRCLE CHAIN: *Let* 1, 2, 3..*be a series of points on a circle. With n points* 1, 2,...,n *associate their centroid* (12..n). *It can then be proved*§ *that*

* It is easily verified that an orthocentre chain coincides with the nine-points chain (16). This is a corollory from the fact that the radii of all these circles is constant. This gives the double Coolidge chain (26).

† These can of course be proved *ab-initio* from the fundamental theorem.

‡ *Vide* Narasinga Rao (26).

§ *Vide* K. O. Augustine (16).

With $n+1$ *points* 1, 2, 3 ... $n+1$, *the* $n+1$ *centroids lie on a circle—the centroidal circle of the* $n+1$ *points.*

With $n+2$ *points the* $n+2$ *centroidal circles* $(12..n+1)$, *etc. are concurrent at a point* $(12..n+2)$.

With $n+3$ *points* 1, 2, ... $n+3$ *the* $n+3$ *points like* $(12..n+2)$ *lie on a circle* $(12..n+3)$. *Since these give four consecutive chains of incidence scheme A, it follows that the chain continues indefinitely.*

If we begin with two points and their centroid, we get the nine-point circle chain.

5.6. THE POINT-O-THEOREM CHAIN stated in 4.42 can again be seen as a consequence of the fundamental Theorem A, since we have four consecutive links verified to be true in the very definitions of (), (i), (ij) and (ijk).

5.7. THE ORIENTED LINE CHAIN: *Let* 1, 2, ..., *be a series of oriented lines. Every triad of these lines* i, j, k *has a unique tangent oriented circle. Let its centre be* (ijk). *Then*

With four lines 1, 2, 3, 4 *the four centres* (123), .., *lie on a circle* (234).

With five lines the five circles like (1234) *are concurrent at a point* (12345) *and so on.*

To get the four consecutive links, start by associating with each of the line i the point (i)* coinciding for all i with P_∞ the point at infinity in the tetracyclic plane. With two lines 1, 2 we can associate the line (12) which is the locus of centres of oriented circles touching 1, 2 properly. Evidently (12) being a line passes through (1) and (2). Also (12) (13) (23) concur at (123) the centre of the unique oriented circle touching 1, 2, 3. By a known

* This does not invalidate the proof, since we can obtain the result by considering the Clifford chain to be the set of lines (12), (13) etc. The essential principles are that the incidences should hold and at the fourth stage, there should be enough elements to fix a circle or a point.

result* the centres (123), (124), (134) and (234) lie on a circle (1234). Thus we have four consecutive links and the entire chain is obtained.

5.8. THE RADICAL AXIS CHAIN[27,28]: Let 1, 2,..., be a series of circles whose centres 1′, 2′,.. lie on a circle C.

(5.82) *With two circles* 1, 2 *we have a radical axis* (12).

(5.83) *With three circles* 1, 2, 3 *we have three radical axes* (12),...*which concur at* (123), *their radical centre.*

(5.84) *With four circles,* 1, 2, 3, 4, *the four points like* (123) *lie on a circle* (1234) *and so on.*

For, with one circle i associate the point $(i) = P_\infty$, the point at infinity. With two circles i, j associate their radical axis (ij) which is evidently incident with (i) and (j). We know that the radical axes (12), (13), (23) concur at (123) and we have to verify 5.84. But this is true, since the angles which (ijk) (kli) subtend at (jkl) and at (lij) are respectively equal to those between $(j'k', k'l')$ and $(l'i', i'j')$ the joins in the former being at right angles to those in the latter; and these angles are supplementary since $i'\ j'\ k'\ l'$ lie on C.

Hence, we have now four consecutive links and the chain therefore continues indefinitely.

5.9. RECTANGULAR HYPERBOLA CHAIN: Let 1, 2,... be a series of rectangular hyperbolas outpolar to a circle (), i.e. having their centres (1), (2),..on ().

(5.92) *With two rectangular hyperbolas* 1, 2, *we can associate a circle* (12)—*the locus of centres of rectangular hyperbolas passing through the intersections of* 1 *and* 2.

(5.93) *With three rectangular hyperbolas* 1, 2, 3, *the three circles* (12), (23), (31) *concur at a point* (123).

(5.94) *With four rectangular hyperbolas* 1,..., 4, *the four points* (123),..., *lie on a circle* (1234) *and so on.*

* *Vide* Steiner[9] and Coolidge[13] p. 357. The result is easily proved by using the angular properties of the cyclic quadrangle.

For, with $p=0$ rectangular hyperbolas, we can associate the circle (). With one rectangular hyperbola i its centre lies on (). With two hyperbolas 1, 2, (12) evidently passes through (1) and (2). Now (12) being the centre-locus of rectangular hyperbolas linearly dependent on 1 and 2, is seen to be the locus of points whose polars w.r.t. the hyperbolas are parallel. So if (12) and (13) meet at (123) other than (1), the polars of (123) w.r.t. (1) and (2) are parallel as also those w.r.t. (1) and (3), and hence also those w.r.t. (2) and (3), i.e. (23) also passes through (123). This proves 5.93. Now we have four consecutive links $p=0$, 1, 2 and 3, and the chain continues indefinitely.

VI Chains derived from the Cox-chain.

6. We derive a number of chains from the Cox-chain by giving different interpretations to the quadric surface in projective three space.

6.1. Narasinga Rao bisector-k-circle chain: Let () be an arbitrary circle of radius k. We shall call such circles k-circles. Let (1), (2),... be circles each of which bisects (). Then

Two circles (1), (2) bisect another unique k-circle, say, (12).

With three circles (1), (2), (3) the three k-circles (12),... are all bisected by a unique circle (123) and so on.

Finally with n circles (1), (2) . . (n) we obtain (12 . .) which is a circle bisecting the k-circles (12 . . $i-1$ $i+1$ . . n) or a k-circle bisected by the circles (12 . . $i-1$ $i+1$ . . n) as n is odd or even.

Proof.* In the usual representation of circles in a tetracyclic plane by points in a projective S_3, it is known

* We give an alternative proof. Consider the following transformation in the inversive or the Euclidean plane. Replace every circle, of radius r, including points by a concentric circle of radius $\sqrt{r^2+k^2}$. It is easy to see that this transformation carries circles into circles and point-circles into k-circles. Also the incidence relation between circle and point is reflected by the property of a circle bisecting a k-circle. It is easy to see, that this transformation applied to the Grace chain 3.6 gives a geometrical procedure to Narasinga Rao's principle of transference from G_0 to G_R. *Vide* (25).

that all k-circles are represented by the quadric $\Omega_k \equiv \Omega - k\pi^2$, where Ω represents the assemblage of point-circles and π the assemblage of straight lines. Thus π is the tangent plane to Ω_k and to Ω at P, the image of the point at infinity. It is easily seen that the condition that a point on Ω_k should be conjugate to some point of S_3 reflects the condition that the k-circle should be bisected by the corresponding general circle. Thus, considering every circle as the assemblage of k-circles it bisects, we can say that every circle will be represented by a plane in S_3, every k-circle by a point on Ω_k; and the necessary and sufficient condition that a circle should bisect a k-circle is that the corresponding plane and point (on Ω_k) should be incident. Then, the chain we have enunciated reduces to chain 3.6, which proves our result. When $k = 0$, this reduces to the Grace chain.

6.2. Let Σ he any quadratic complex of circles, i.e. circles represented in S_3 by points on a quadric Σ. Let Ω be the quadric in S_3 which represents the assemblage of point-circles. Then any circle can be represented indifferently by a point or its polar plane w.r.t. Ω. The condition that two circles be orthogonal is that the plane representing either of these circles be incident with the point representing the other. We shall refer to circles of the quadratic complex as σ-circles.

Now start with a point (1) on Σ and draw planes (1), (2),..through (1). On the line of intersection of the planes (i), (j) take the point (ij), its other intersection with Σ. Let (ijk) be the plane through (ij), (jk) and (ki). Then, by the Cox theorem, the planes (ijk), (jkl), (kli), (lij) concur at $(ijkl)$ on Σ and so on. Translating these into tetracyclic geometry, we have:

6.3. THE σ-CIRCLE CHAIN. Let () be σ-circle; let (1) (2)..be circles orthogonal thereto.

With two circles (1) (2), *there is a unique other σ-circle* (12) *orthogonal to* (1) *and* (2).

With three circles (1) (2) (3), *the three σ-circles* (12), (23), (31) *are orthogonal to a unique circle* (123).

Finally, with n circles we obtain (12... *n*) *which is or is not a circle according as n is even or odd.*

If, in the above, we take Σ to be the assemblage of *k*-circles, we obtain

6.31. NARASINGA RAO'S ORTHOGONAL EXTENSION OF THE GRACE CHAIN OF THEOREMS*. Let () be a *k*-circle; (1), (2),.. be circles orthogonal to ().

Two circles (1) (2) *have another unique k-circle* (12).

With three circles (1), (2), (3) *the three k-circles* (12) (23) (31) *are orthogonal to a circle* (23).

With four circles (1) (2) (3) (4), *the four circles like* (123) *are orthogonal to a k-circle* (1234) *and so on.*

6.311. In the special case when $a = 0$ we obtain the Grace chain of theorems.

6.4. Let every line-element of an oriented circle† be rotated both ways through a fixed angle α. The resulting aggregate of elements is called an oriented turbine. An oriented circle is a special turbine for which $\alpha = 0$. We shall say that a circle touches a given turbine T if they have a line-element in common. We shall call such circles T-circles.

It is known that all turbines in a plane can be represented by points (or dually by primes) in a projective

* This chain can be obtained by the following transformation from the Grace-chain. Replace every point by a *k*-circle with centre at the point; and every circle of radius '*r*' by a concentric circle of radius $\sqrt{r^2-k^2}$. This transformation establishes a principle of duality between the geometry of points and circles and the geometry of *k*-circles and circles. The transform of the Grace chain in this process gives 6.3. This transformation gives the geometrical significane of Dr. Narasinga Rao's principle of duality.

† In the remaining part of this section, all the lines, circles, line-elements and turbines are oriented. A circle and a line (circle) 'touch' if they have proper contact.

S_4. The points (primes) representing oriented circle will then lie on (envelope) a quadric Ω. The condition that a turbine and a circle touch is then reflected by the incidence of the prime corresponding to the turbine with the point on Ω corresponding to the circle (or dually). Thus all T-circles are represented by points on the section Ω' of Ω by the prime T and hence may be considered as the envelope of the k-circles. Now the point and prime corresponding to a turbine T define a collineation in S_4 leaving Ω invariant. It is easy to see that a pair of conjugate points on Ω are transformed into another pair of conjugate points; and that points on the section Ω' of Ω by T are transformed into themselves. Correspondingly we have in the inversive plane, the

6.41. T-INVOLUTION: Every turbine T gives rise to a T-involution between circles (and more generally turbines) in the plane, the mate C' of any circle C in the involution being the envelope of T-circles touching C. We shall call C, C' an associated pair of circles.

Now, since every tangent prime to Ω' meets T in a unique plane, and through any plane in T there pass a unique pair of tangent planes to Ω' which (as well as the corresponding points of contact) represent a pair of associated circles C, C' in the T-involution and since points on Ω' are represented by T-circles, we can say that we have a representation of associated circle-pairs in the plane by means of points and plane in an S_3, viz. T. Any T-circle is represented by a point on a quadric Ω and any general circle (along with its associate) by planes in T. A circle and a T-circle touch if the corresponding plane and point are incident. Hence, by considering a Cox-configuration on Ω' we get the following

6.42. THEOREM. Let (1) be at a T-circle. Let (1), (2),...be a series of *oriented* circles touching ().

Two circles (1), (2) *touch another unique T-circle* (12).

With three circles (1) (2) (3), *the three T-circles* (12) (23) (31) *have two common tangent circles* (*associcated*) (123) *and* (123)′.

With four circles (1)...(4) *the four pairs of associated circles like* (123), (123)′ *touch a T-circle* (1234) ; *and so on.*

With an even number of circles (1)...(n) *we have a unique T-circle* (12..n) *and with an odd number of circles we have a pair of associated circles* (12..n), (12..n)′.

Three particular cases of the above are worth noting. First, when T coincides with the turbine invariant for G_k geometry (in which the totality of k-circles is invariant) all T-circles become k-circles, i.e. oriented circles of radius k. Here the associated circles have radius $k-r$, $k+r$ and are concertric. Thus, we can fix upon circles of radius $> k$ alone and then we have the

6.421. THEOREM. *Let* () *be a k-circle and let* (1) (2) (3).. *be circles of radius* $> k$, *tangent to* ().

Two circles (1) (2) *have another unique tangent k-circle* (12).

With three circles (1) (2) (3), *the three k-circles* (12) (23) (31) *touch a unique circle of radius* $> k$ (*say*) (123), (*and its associated circle* (123)′).

With four circles (1) (2) (3) (4), *the four circle pairs* (123) (123)′, *etc. touch a unique k-circle* (1234) *and so on.*

When $k \to \infty$, i.e. when the turbine T becomes the invariant of Laguerre geometry (i.e.) the corresponding point lies on Ω', the associated circle of *every* circle becomes L and hence this associate may conveniently be omitted from consideration. Thus, we have

6.4211. GRACE-CHAIN IN LAGUERRE GEOMETRY:(13)

Let (1) (2)...be oriented circles tangent to an oriented line ().

Two circles (1) (2) *have a unique common tangent line* (12).

With three circles (1) (2) (3), *the three lines* (12) (23) (31) *touch a unique circle* (123).

With four circles (1)...(4) *the four circles* (123)...*touch a common line* (1234) *and so on.*

The next noteworthy case is when T degenerates into an oriented circle* O. Then, the associated circle of every circle is O itself and thus, we have:

6.422. THEOREM. *Let () be an O-† circle, and let (1) (2)..be circles tangent to (). Two circles (1) (2) have a unique tangent O-circle (12) and so on.*

With n circles (1) (2)..(n) we arrive at (12...n) which is or is not an even circle according as n is even or odd.

When the circle O becomes the point-circle L of Laguerre geometry we once again get 6.421 from 6.42. Thus 6.421 and 6.422 can be considered to be two extensions of the analogue of the Grace chain in Laguerre geometry.

6.43. Again, when T coincides with the invariant of Möbius geometry, the associated circle of every circle is the same circle with orientation reversed, and thus at each stage we can replace associated circle-pairs by a circle oriented both ways, i.e. by a non-oriented circle. Also, the T-circles become points, and this reduces the T-circle chain to the Grace chain itself.

Likewise, we could have applied the dual Cox-chain in 6.42 (or the earlier chain) and obtained a series of chains like 6.421, 6.422 etc. We state only the one corresponding to 6.4211.

6.5. THE DUAL COOLIDGE CHAIN[13]: *Let (1), (2),..., be oriented lines tangent to a circle (). Let (ij) be an arbitrary circle touching (i) (j) and such that the circles (ij), (jk), (ki) have a common tangent line (ijk). Then,*

With four lines (1),...,(4) the four lines like (123) touch a circle (123) and so on.

Just as we obtained the point-O-chain from the dual Grace chain, we obtain as a particular relation of 6.5 the

* In this case Ω' will become a quadric cone in S_3. But this does not invalidate the result, since the Cox chain holds even on a degenerate quadric.

† Circle (oriented) touching a given circle O.

6.51. COOLIDGE POINT-O-CHAIN IN LAGUERRE GEOMETRY[13]. *Let* $(1)' \ldots (n)'$ *be oriented circles touching another circle* $()'$ *and let* $(1) \ldots (n)$ *be tangents to a circle* $()$ *and touching* $(1)' \ldots (n)'$ *respectively. Then with these 2 n circles we can associate a line or a circle* $(12..)$ *according as n is odd or even such that the derived elements follow incidence Scheme A.*

To start the chain, we have only to associate when $n=2$, the circle (12) touching (1) (2) and the common tangent other than $()$ of $(1)'$ and $(2)'$.

6.6. A CHAIN FOR LINES AND QUADRICS OF A CONGRUENCE IN PROJECTIVE S_3: *Let* L L' *be two given lines and* $()$ *a transversal to these. Consider a series of quadrics* $(1), \ldots$ *containing* L, L' *and* $()$. *Every pair of these quadrics* (1), (2) *have a common generator* (12) *meeting* L *and* L'.

Given three quadrics (1), (2), (3), *the three lines* (12) (13) (23) *lie on a quadric* (123) *containing* L, L'.

With four quadrics $(1) \ldots (4)$, *the four quadrics like* (123) *have a common generator* (1234) *meeting* L, L'; *and so on.*

For, in the representation of lines in a plane by points on a quadric in S_5 and of line complexes by points not on the quadric or by its polar prime, it is known that lines of a linear congruence will be represented by points on a quadric in S_3 and reguli composed of lines of the congruence by a plane in S_3 and that the condition of incidence of a line of the congruence with a regulus (of the congruence) is that the corresponding point and plane be incident. Applying Cox chain to this representative S_3, we obtain the above where L, L' are the directrices of the congruence.

VII OTHER CHAINS.

7. In proving the fundamental theorem 4, the only properties of points and circles that have been used are:

7.1. Two 'circles' with one common point have one and only one other common point.

7.2. Three "points" always lie on a circle and determine it uniquely.

7.3. The four-circle theorem in the Grace chain (or its dual) is true.

Hence, we can expect the chain of theorems to hold for any two types of elements 'α' or circles and 'β' or 'points' connected by a mutual binary relation 'incidence' and satisfying conditions given above.

All the results in the last section can be considered from this point of view, by showing that for the types of elements considered*, the above three conditions are valid. Hence we can deduce the chains just as we derived the Grace chain, etc.

We shall give below two other different examples:

Given a non-oriented line l in a 3-space, and an oriented line o, we obtain by giving o different screw motions about l, a system of ∞^2 lines. These have the property that a unit force along any one of these has the same resolved part in the direction of l and also the same moment about l. We shall call this system an 'axial congruence' or simply a 'congruence' and the line l as the axis of the congruence.

It is then easy to verify, that two congruences with a common line have another line in common, that three oriented lines always belong to and determine a unique congruence(18) and further that congruences through an oriented line satisfy the analogue of the four-link theorem in the Grace chain. This gives, by repeated applications of the four-link theorem as the fundamental theorems the following

* Oriented circles and oriented lines, oriented circles and k-circles with orthogonality, or the bisection of the k-circle as the 'incidence' relation, or circles and T-circles with tangency as the relation or quadrics and lines of a congruence with the line being a generator of the quadric as the incidence relation.

4

7.4. RICHMOND CHAIN OF THEOREMS FOR LINES IN THREE SPACE[19]: *Let* (1), (2),.., *be a series of congruences through an oriented line* ().

Two congruences (1), (2) *have a unique oriented line* (12).

With three congruences (1), (2), (3), *the three oriented lines* (12), (23), (31) *belong to a unique congruence* (123).

With four congruences (1),..., (4) *these contain a unique common oriented line* (1234), *and so on.*

In the special case, when (1), (2) are right circular cylinders (when the generating lines *o* are parallel to *l*), it is seen that all the congruences in the chain reduce to right circular cylinders. The axes of all these are parallel and the section by any plane perpendicular to this diameter gives the Grace chain. As another example, we need only mention

7.5. NARASINGA RAO'S MIQUEL-CLIFFORD-CHAIN IN LINE ELEMENT GEOMETRY: The conditions 7.1, 7.2 and 7.3 are easily verified to be true (by using a method similar to Narasinga Rao's method for the earlier stages in the same), whence by the fundamental theorem, the entire configuration can be obtained. We simply state the configuration finally obtained in the nth stage.

For every positive value of n, *it is possible to find systems of* 2^n *c-conics and* 2^n *congruence* lines, with the property that*

(*i*) *every congruence line is met exactly by* $n+1$ *c-conics and*

(*ii*) *every c-conic is met exactly by* $n+1$ *congruence lines.*

8. CONCLUSION. We saw that a chain following the incidence Scheme A' (or A) which is true for four consecutive links is true indefinitely, and has its configuration made up of a number of overlapping Cox (Grace) configurations; and we derive as particular cases a number of chains in Euclidean, Möbius, Laguerre and Narasinga Rao G_k geometries, and in line-geometry. It is characteristic

* For definition of these and for proving 7.1, 7.2 and 7.3 as indicated, see [26].

of these chains, that we start with one type of element, and increase the number of these elements taken by unity at every stage; and derive alternately two other types of elements with a mutual 'incidence' relation. Typical of all these, and giving rise to a large number of chains as concrete realisations thereof, is the projective chain of theorems due to Cox, which however, as Richmond observes, have not received the attention which they deserve.

8.1. POSTSCRIPT. We have seen that if it is possible to associate with n elements, $n=p, p+1, p+2$ or $p+3$, a circle (point) satisfying incidence Scheme A, then it is possible to continue the association for all $n>p$. It is easily seen likewise that the same association is possible for all n, even when $0 \leqslant n < p$.

Thus the general method of associating a circle (point) with any n elements of a set in such a way that a chain of incident circles and points may ensue is as follows. Associate with every element a circle passing through a fixed point, and associate with n elements the circle (point) obtained as the final result of the Grace chain generated by n concurrent circles corresponding to the n elements (or dually).

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ON DIFFERENCE-SETS.

BY

S. CHOWLA.

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1. The positive integers $d_1, \ldots, d_t$ incongruent (mod m) are said to form a difference-set (mod m ; $m_1, \ldots, m_r$), where $m_1, \ldots, m_r$ are any factors of m, if the number of solutions of the congruence g congruent to $d_i - d_j$ (mod m) (i, j taking values from 1 to t both inclusive), is independent of g, whenever g is not a multiple of $m_1, \ldots, m_r$. We call the set as a difference-set of t integers (mod m ; $m_1, \ldots, m_r$).

Ex. 1. There exists a difference-set of 12 integers (mod 35 ; 5, 7), viz.

S: 1, 3, 4, 9, 11, 12, 13, 16, 17, 27, 29, 33.

It is easily verified that every integer prime to 5 and 7 can be represented exactly 4 times as a difference (mod 35) of two numbers of S. Thus, we have

$$1 = 4-3 = 12-11 = 17-16 = 13-12$$
$$11 = 9-33 = 12-1 = 3-27 = 27-16$$

the equality sign meaning 'congruent (mod 35)'.

This example is a special case of the following general

THEOREM. *Let m be quadratfrei and congruent to* 3 (mod 4), $m = m_1 \ldots m_r$, *where the m_r's are distinct primes. Let $\phi(m)$ denote the number of positive integers not exceeding m and prime to it. Then there exists a difference-set of $\frac{1}{2}\phi(m)$ integers* (mod m; $m_1, \ldots, m_r$), *consisting of these $\frac{1}{2}\phi(m)$ integers x lying between* 0 *and m and satisfying $(x/m) = +1$.*

Here (x/m) is the extended Legendre-Jacobi symbol, defined as the product of $(x/m_1), \ldots, (x/m_r)$, each of the latter being a Legendre symbol of quadratic residuacity.

The special case $r = 1$, i.e. m, a prime, is due to R. C. Bose*.

We note that S is invariant for the transformation $x' = 4x$ (mod 35); for, multiplying the numbers by 4 and reducing (mod 35), we get

S': 4, 12, 16, 1, 9, 13, 17, 29, 33, 3, 11, 27,

which is the same as S, the numbers being in a different order.

CONJECTURE 1. *Every difference-set* (mod m) *is invariant under a certain non-trivial linear transformation.*

By trivial linear transformations, we mean transformations $x' = ax+b$ (mod m) in which a is congruent to $+1$ or -1 (mod m).

Ex. 2.† The numbers S: 1, 4, 9, 11, 24 form a difference-set (mod 24; 6). This is invariant under the transformation $x' = 5x+4$ (mod 24), for it then becomes S': 9, 24, 1, 11, 4.

2. We shall call the difference-set

S: $d_1, \ldots, d_t$ (mod m; $m_1, \ldots, m_r$)

equivalent to the difference-set

S': $ad_1+b, \ldots, ad_t+b$ (mod m; $m_1, \ldots, m_r$)

* *Annals of Eugenics*, 9.

† Special case $m = 5$ of a theorem of R. C. Bose, *Jour. Ind. Math. Soc.* (2), 6 (1942), 1-15.

whenever a is prime to m. For, if S represents a difference-set mod $(m; m_1, \ldots m_r)$, so does S' for all a and b, provided that a is prime to m.

Given t and $m, m_1, \ldots, m_r$ how many non-equivalent difference-sets of t integers (mod $m; m_1, \ldots, m_r$) can be found? The author has not been able to construct a single example of two non-equivalent sets whose parameters $t, m, m_1 \ldots$ are common.

Ex. 3. α: 1, 2, 4; β: 1, 4, 3 are both examples of difference-sets (mod 7; 7); but β can be derived from α by the transformation $x' = 3x+5$ (mod 7).

Ex. 4. The numbers 1, 2, 4, 8, 16, 32, 37, 55, 64 form a difference-set (mod 73, 73). I do not know of any other difference-set of 9 numbers (mod 73; 73).

CONJECTURE 2. *It is impossible to construct two non-equivalent difference-sets of t integers* (mod m; $m_1, \ldots, m_r$), *where t, m, $m_1 \ldots$ are given numbers.*

I have proved an extension of a theorem of Bose that

If p is a prime we can find a difference-set of p integers (mod p^2-1; $p+1$), *which is invariant for the transformation* $x' = px$ (mod p^2-1).

Similarly we can prove an extension of a theorem of James Singer that

If p is a prime and is congruent to 2 (mod 3) *there exists a difference-set of $p+1$ integers* (mod p^2+p+1; p^2+p+1) *which is at the same time invariant for the transformation* $x' = px$ (mod p^2+p+1).

These results support Conjecture 1.

3. Using a process described by Bose* we can construct 20 difference-sets of 5 integers (mod 24; 6) from the *row-number I* of the Latin squares of order 25 for all possible canonical sets—here 4 in number. Each canonical set gives rise to 5 difference-sets. We shall enumerate these sets and show that they are all equivalent, thus confirming Conjecture 2.

* Also, Bose and Nair, *Sankya*, (1941), 381.

I. (*i*) x: 1, 4, 24, 9, 11
(*ii*) y: 1, 10, 5, 12, 2
(*iii*) z: 1, 18, 23, 14, 15
(*iv*) u: 1, 6, 22, 21, 8
(*v*) v: 1, 16, 20, 17, 3.

The following linear transformations of the sets among themselves are obvious, 24 being the modulus of transformation.

$y = x+1$, $z = y+13$, $u = z+7$, $v = u-5$.

II. (*i*) x: 1, 18, 22, 23, 9
(*ii*) y: 1, 6, 5, 16, 8
(*iii*) z: 1, 3, 20, 11, 24
(*iv*) u: 1, 12, 2, 21, 4
(*v*) v: 1, 14, 10, 15, 17.

We have $y = x+7$, $z = y-5$, $u = z+1$, $v = u+13$. The transformation of II are a cyclic transformation of those of I.

III. (*i*) x: 1, 9, 11, 16, 12
(*ii*) y: 1, 22, 5, 24, 14
(*iii*) z: 1, 2, 15, 6, 23
(*iv*) u: 1, 18, 10, 21, 20
(*v*) v: 1, 17, 3, 4, 8.

We have $y = x+13$, $z = y+1$, $u = z-5$, $v = u+7$.

IV. (*i*) x: 1, 23, 9, 6, 10
(*ii*) y: 1, 18, 5, 4, 20
(*iii*) z: 1, 11, 12, 3, 8
(*iv*) u: 1, 24, 14, 21, 16
(*v*) v: 1, 15, 17, 2, 22.

We have $y = x-5$, $z = y+7$, $u = z+13$, $v = u+1$.

The transformations of IV can be got from those of III by a cyclic transformation of 13, 1, —5, 7.

To prove that all these 20 difference-sets are equivalent it is enough if we show that the first rows of each of the 4 sets are equivalent. Denoting these by x_1, x_2, x_3, x_4 we have $x_2 = -7x_1 - 8$, $x_3 = -11x_1 - 12$, $x_4 = -5x_1 + 6$ (mod 24).

LIBRARY

SOME GENERALIZATIONS OF THE DIVISOR FUNCTION.

BY

P. KESAVA MENON.

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1. Let $\tau(M_1, \ldots, M_r)$ represent the number of sets of divisors $d_1, \ldots, d_r$ of $M_1, \ldots, M_r$ respectively such that the greatest common divisor of $d_1, \ldots, d_r$ is unity. We shall prove that $\tau(M_1, \ldots, M_r)$ is multiplicative, i.e. it satisfies the relation

$$\tau(M_1N_1, \ldots, M_rN_r) = \tau(M_1, \ldots, M_r)\ \tau(N_1, \ldots, N_r),$$

where ΠM_i is prime to ΠN_i.

PROOF. Let ΠM_i be prime to ΠN_i. A divisor of M_iN_i can be uniquely written in the form $d_i\delta_i$, where d_i, δ_i are divisors of M_i, N_i respectively. Obviously Πd_i is prime to $\Pi\delta_i$. Therefore, if $g\,(M_1, \ldots, M_r)$ denotes the g.c.d. of $M_1, \ldots, M_r$, then $g(d_1\delta_1, \ldots, d_r\delta_r) = g(d_1, \ldots, d_r)\ g(\delta_1, \ldots, \delta_r)$. It follows that if $g(d_1\delta_1, \ldots, d_r\delta_r) = 1$, then $g(d_1, \ldots, d_r) = g(\delta_1, \ldots, \delta_r) = 1$, and vice-versa. This gives the required result.

2. Let $\prod_p p^{a_i}$ be the resolution into prime factors of M_i $(i = 1, \ldots, r)$. Then by the multiplicative property of $\tau(M_1, \ldots, M_r)$,

$$\tau(M_1, \ldots, M_r) = \prod_p \tau(p^{a_1}, \ldots, p^{a_r}).$$

Also, $\tau(p^{a_1}, \ldots, p^{a_r}) = (a_1+1) \ldots (a_r+1) - a_1 \ldots a_r$.

Hence, we obtain

$$\tau(M_1, \ldots, M_r) = \prod_{i=1}^{n} \tau(M_i) \prod_p \left\{1 - \frac{a_1 \ldots a_r}{(a_1+1) \ldots (a_r+1)}\right\}.$$

If $\tau_r(M)$ denotes the number of sets of r divisors of M whose g.c.d. is unity,

$$\tau(M) = \tau(M, \ldots, M) = \left\{\tau(M)\right\}^r \prod_p \left\{1 - \left(\frac{a}{a+1}\right)^r\right\}.$$

2. Let $E(M) = 1$ for all values of M, and $E_0(M) = 1$ if $M = 1$, and 0 otherwise. Also let $E(M_1,\ldots, M_r) = \prod_{i=1}^{r} E(M_i)$ and $E_0(M_1,\ldots, M_r) = \prod_{i=1}^{r} E_0(M_i)$.

The expression

$$\Sigma f(d_1,\ldots,d_r)\phi(M_1/d_1,\ldots,M_r/d_r)$$

summed for all divisors d_i of M_i $(i = 1, 2,\ldots,r)$ is called the composite[1] of f and ϕ and is denoted by $(f.\phi)(M_1,\ldots,M_r)$. $(f.f)$ is denoted by f^2, $(f^2.f)$ by f^3 and so on. f has a unique inverse f^{-1}, which is such that $f.f^{-1} = E_0$ (provided $f(1,\ldots,1) \neq 0$, which is true for all multiplicative functions).[1]

$f(M_1,\ldots,M_r)$ is called a principal function[1] of r arguments equivalent to $\theta(M)$ if $f(M,\ldots,M) = 0$ unless $M_1 = \ldots = M_r$, and $f(M,\ldots, M) = \theta(M)$. We shall then write $f = \text{princ}\ \theta$.

From the definition of $E_0(M)$ it follows that $E_0(g(M_1\ldots,M_r))$ is a function of r arguments which is equal to unity if the g. c. d of the arguments is unity and zero otherwise. It is easily seen that $\tau(M_1,\ldots,M_r)$ is the composite of $E(M_1,\ldots,M_r)$ and $E_0(g(M_1,\ldots,M_r))$. By a known theorem[1],[2] on multiplicative functions which are functions of the g. c. d. of the arguments we see that

$$E_0(g(M_1,\ldots,M_r)) = (E.\ \text{princ}\ E^{-1})(M_1,\ldots,M_r).$$

Therefore $\tau(M_1,\ldots,M_r) = (E^2.\ \text{princ}\ E^{-1})\ (M_1,\ldots,M_r)$.

Now $E^2(M_1,\ldots,M_r) = \prod_{i=1}^{r} E^2(M_i)$, where $E^2(M)$ is the divisor function $\tau(M)$ and $E^{-1}(M)$ is the Möbius function $\mu(M)$. Hence the above relations may be written as

$$\tau(M_1,\ldots,M_r) = \Sigma\mu(d)\tau(M_1/d)\ldots\tau(M_r/d),$$

where the summation is for all common divisors d of $M_1,\ldots, M_r$.

Composing both sides by princ E we get the equivalent result

$$\Sigma\tau(M_1/d,\ldots,M_r/d) = \prod_{i=1}^{r} \tau(M_i)$$

the summation being for all common divisors d of $M_1, \ldots, M_r$.

COR. $\Sigma \tau_r(d) = \left\{ \tau(M) \right\}^r,$

the summation being for all divisors d of M.

2.1. Let $\tau'(M_1, \ldots, M_r)$ represent the number of sets of divisors $d_1, \ldots, d_r$ of $M_1, \ldots, M_r$ respectively every two of the d's being mutually prime. Then $\tau'(M_1, \ldots, M_r)$ is a multiplicative function. The proof is as in §1.1.

2.2. If $M_i = \prod_p p^{a_i}$ $(i = 1, \ldots r)$, then

$$\tau'(M_1, \ldots, M_r) = \prod_p \tau'(p^{a_1}, \ldots p^{a_r})$$

$$= \prod_p (1 + a_1 + \ldots + a_r) = \tau(M_1 M_2 \ldots M_r).$$

$$\tau_r'(M) = \tau'(M, M, \ldots, M) = \prod_p (1 + ra), \; M = \prod_p p^a.$$

2.3. More generally, the function representing the number of sets of divisors $d_1, \ldots, d_r$ of $M_1, \ldots, M_r$ every s of which has unity for g. c. d. is a multiplicative function. If $\tau_s'(M_1, \ldots, M_r)$ denotes this function we can show that

$$\tau_s'(M_1, \ldots, M_r) = \prod_p \left\{ 1 + \Sigma_1 + \ldots + \Sigma_{s-1} \right\},$$

where Σ_i denotes the sum of the homogeneous products i at a time of $a_1, \ldots, a_r$ and $M_i = \prod_p p^{a_i}$.

COR. *If $\tau'_{s,r}(M)$ denotes the function representing the number of sets of r divisors of M every s of which has unity for g.c.d. then*

$$\tau'_{s,r}(M) = \prod_p \left\{ 1 + \binom{r}{1} a + \ldots + \binom{r}{s-1} a^{s-1} \right\},$$

where $M = \prod_p p^a$.

3.1. Let $\tau''(M_1, \ldots, M_r)$ represent the number of sets of r divisors $d_1, \ldots, d_r$ of $M_1, \ldots, M_r$ such that $g(d_1, \ldots, d_r) = g(M_1/d_1, \ldots, M_r/d_r) = 1$. Then it can be shown that τ'' is multiplicative. Also $\tau''(M_1, \ldots, M_r) = \left\{ \prod_{i=1}^{r} \tau(M_i) \right\}$

$$\times \prod \left\{ 1 - \frac{2a_1 \ldots a_r - (a_1 - 1) \ldots (a_r - 1)}{(a_1 + 1) \ldots (a_r + 1)} \right\}$$

the second product being for all common prime factors of $M_1, \ldots, M_r$. Writing $M_1 = \ldots = M_r = M$ we see that

if $\tau_r''(M)$ denotes the number of sets of r divisors of M whose g.c.d. as well as the g.c.d. of their complimentary factors are both unity, then

$$\tau_r''(M) = \prod_p \left\{ (a+1)^r - 2a^r + (a-1)^r \right\},$$

where $M = \Pi p^a$.

3.2. It can be seen that $\tau''(M_1, \ldots, M_r)$ is the composite of $E_0[g(M_1, \ldots, M_r)]$ with itself. In symbols—

$$\tau''(M_1, \ldots, M_r) = (E_0(g).(E_0(g))\,(M_1, \ldots, M_r)$$
$$= E.\text{princ}E^{-1}.\,E.\text{princ}E^{-1})\,(M_1, \ldots, M_r)^{(1),(2)}$$
$$= (E^2.\,\text{princ}E^{-2})\;(M_1, \ldots, M_r),$$

This may be expressed in the two forms

$$\tau''(M_1, \ldots, M_r) = (E.E^{-4}(g))\;(M_1, \ldots, M_r)^{(1)\,(2)}$$
$$= \Sigma\mu(g(d_1, \ldots, d_r)),$$

where the summation is for all divisors d_i of M_i, $i = 1, \ldots r$, and

$$\tau''.\;\text{princ}\,E^2 = E^2,$$

i.e. $\Sigma\tau''\;(M_1/d, \ldots, M_r/d)\,\tau(d) = \prod_{i=1}^{r}\;\tau(M_i)$

the summation being for all common divisors d of $M_1, \ldots, M_r$.

Cor. *Writing $M_1 = \ldots = M_r = M$ we get*

$$\Sigma\tau_r''(M/d)\,\tau(d) = \left\{ \tau(M) \right\}^r,$$

where the summation is for all divisor, d of M.

4.1. Let $\tau'''(M_1, \ldots, M_r)$ denote the number of sets of divisors $d_1, \ldots, d_r$ of $M_1, \ldots, M_r$ which satisfy the condition $g(d_1, \ldots, d_r,\; M_1/d_1, \ldots, M_r/d_r) = 1$. Then τ''' can be shown to be multiplicative. Hence ,

$$\tau'''(M_1, \ldots, M_r) = \prod_p \tau'''\;(p^{a_1}, \ldots, p^{a_r}),$$

where $M_i = \prod_p p^{a_i}$ $(i = 1, \ldots, r)$. But

$$\tau'''(p^{a_1}, \ldots, p^{a_r}) = (a_1+1)\ldots(a_r+1) - (a_1-1)\ldots(a_r-1)$$
$$\text{if } a_i > 0\;(i = 1, 2, \ldots r),$$
$$= (a_1+1)\ldots(a_{i-1}+1)(a_{i+1}+1)\ldots(a_r+1)\text{ if } a_i = 0.$$

Therefore

$$\tau'''(M_1, \ldots, M_r) = \prod_{i=1}^{r}\tau(M_i)\,\Pi\left\{1 - \frac{(a_1-1)\ldots(a_r-1)}{(a_1+1)\ldots(a_r+1)}\right\}$$

the second product being for all common prime factors of $M_1, \ldots, M_r$.

In particular if we denote the number of sets of r divisors $d_1, \ldots, d_r$ of M which satisfy the condition in 4.1, by $\tau_r'''(M)$, then

$$\tau_r'''(M) = \left\{\tau(M)\right\}^r \prod_p \left\{1 + \left(\frac{a-1}{a+1}\right)^r\right\}, \quad M = \prod_p p^a.$$

4.2. It can be shown that

$$\tau'''(M_1, \ldots, M_r) = \Sigma\tau(M_1/d, \ldots, M_r/d),$$

where τ is the function defined in §1 and the summation is for all quadratfrei common divisors d of $M_1, \ldots, M_r$. In other words $\tau'''(M_1, \ldots, M_r)$ is equal to the number of sets of r divisors $d_1, \ldots, d_r$ of $M_1, \ldots, M_r$ whose g.c.d. is quadratfrei. This may be verified by first showing that the latter function is multiplicative and then by evaluating it when the arguments are powers of the same prime.

5. Let $\tau^{(iv)}(M_1, \ldots, M_r)$ denote the number of sets of divisors $d_1, \ldots, d_r$ of $M_1, \ldots, M_r$ such that $g(d_1, \ldots, d_r) = g(M_1, \ldots, M_r)$. Then $\tau^{(iv)}$ is multiplicative, the proof being similar to §1.1.

Hence $\tau^{(iv)}(M_1, \ldots, M_r) = \prod_p (a_1+1-a)\ldots(a_r+1-a)$,

where $a = \min.(a_1, \ldots, a_r)$ and $M_i = \Pi\, p^{a_i}$. Therefore also

$$\tau^{(iv)}(M_1, \ldots, M_r) = \prod_{i=1}^{r} \tau(M_i/M),$$

where $M = g(M_1, \ldots, M_r)$.

6. Finally, if $\tau^{(v)}(M_1, \ldots, M_r)$ denotes the number of sets of divisors $d_1, \ldots, d_r$ of $M_1, \ldots, M_r$ such that the l.c.m. of $d_1, \ldots, d_r$ is equal to the l.c.m. of $M_1, \ldots, M_r$ then we may show similarly that $\tau^{(v)}$ is multiplicative and that

$$\tau^{(v)}(M_1, \ldots, M_r) = \left\{\prod_{i=1}^{r} \tau(M_i)\right\} \Big/ \tau(M),$$

where M is the l.c.m. of $M_1, \ldots, M_r$.

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MADRAS

NOTES ON GROUP THEORY VII

BY

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VII. THE IDEMPOTENT RESIDUE CLASSES AND THE MAPPINGS $\{m\}$

In the first of these notes, the conditions have been discussed for a mapping $\{m\}$ — i.e. the representation of each element a of a group G by a^m—to be an endomorphism. Denoting by $E(G)$ the set of numbers m for which $\{m\}$ is an endomorphism and by $M = M(G)$ the smallest positive number out of $E(G)$ which maps G on an Abelian subgroup, say $A(G)$, then the elements of $E(G)$ were shown to form full classes of residues (mod M) satisfying certain conditions. In the present paper it will be shown that those necessary conditions are also sufficient and a full analysis of the admissible sets $E(G)$ will be given. The investigation, which affords the consideration of certain semigroups of idempotent residue classes (mod M), leads to the following results :

THEOREM 1. *$E(G)$ consists of 2^t idempotent classes of residues* (mod M), *namely all those numbers which are congruent* 0 *or* 1 (mod q_j), *for $j = 1,\ldots,t = t(G)$; the q's are all > 2, they are relatively prime one to another and*

$$q_1 \ldots q_t = M. \qquad (1)$$

If $A(G) = 1$, G is said to belong to $q_1,\ldots,q_t$.

THEOREM 2. *If G belongs to $q_1,\ldots,q_t$, then $G = H_1 \times \ldots \times H_t$, where the H's belong to the corresponding q's.*

THEOREM 3. *If $q_1,\ldots,q_t$ are relatively prime and > 2, then there exists a finite group belonging to $q_1,\ldots,q_t$.*

1. Let M be any integral number, the condition $m(1-m)\equiv 0 \pmod M$ means that $\pmod M$, the class (m) and therefore the class $(1-m)$ are idempotent. Put $(m, M) = r$ and $(1-m, M) = s$, then $rs = M$ and

$$\begin{aligned} m &\equiv 0 \pmod r, & m &\equiv 1 \pmod s \\ 1-m &\equiv 1 \pmod r, & 1-m &\equiv 0 \pmod s. \end{aligned} \tag{2}$$

On the other hand, given r and s with $rs = M$ and $(r,s) = 1$, the congruences (2) determine the idempotent classes (m) and $(1-m)$ uniquely. Denote by B any non-empty semigroup of idempotent residue classes $\pmod M$ which has the property that when (m) belongs to B, then $(1-m)$ also belongs to B. This semigroup must necessarily contain $(m[1-m]) = (0)$ and therefore (1). The necessary conditions for $E(G)$ established in Note I can be enunciated as follows: *For $M = M(G)$, the set $E(G)$ is a system B not containing any pair of consecutive classes except (0), (1).*

Let (m) and (m') belong to B, then there exist 4 numbers r_1, r_2, s_1, s_2 such that $m\equiv 0 \pmod{r_1 r_2}$, $m \equiv 1 \pmod{s_1 s_2}$, $m' \equiv 0 \pmod{r_1 s_2}$, $m' \equiv 1 \pmod{r_2 s_1}$ and the corresponding congruences for $1-m$ and $1-m'$ hold. Now (mm'), $(m[1-m'])$, etc. are idempotent classes contained in B; moreover $mm' \equiv 0 \pmod{r_1 r_2 s_1}$, $mm' \equiv 1 \pmod{s_2}$, and similar congruences for the other products hold. Hence B contains all those residue classes which are congruent to 0 or 1 $\pmod{r_1}$, $\pmod{r_2}$, $\pmod{s_1}$, $\pmod{s_2}$; on the other hand, the numbers satisfying these congruences are forming idempotent residue classes mod M which constitute a semigroup of type B. If any one of the numbers r_1, r_2, s_1, s_2 is equal to 1, it may be omitted. Similarly if any set of integers $w_1, \ldots, w_u$ which are > 1 and relatively prime, is given and $w_1 \ldots w_u = M$, then the numbers which are congruent 0 or 1 to the w's are forming full classes of residues $\pmod M$ and these classes constitute a semigroup B' of type B. Suppose now that B contains B' as well as an idempotent class (m), where

$m \equiv 0 \pmod r$, $m \equiv 1 \pmod s$ and $rs = M$. Put $v_1,\ldots,v_p$ for the highest common factors $\neq 1$ of r and s with the w's, then B must contain all the residue classes of M which are congruent to 0 or 1 modulo the v's and $v_1\ldots v_p = M$. As B contains only a finite number of residue classes, it results by mathematical induction that there exists a factorization of M into factors q_j which are > 1 and relatively prime, $M = q_1\ldots q_t$, such that B consists of those and only those residue classes which are congruent to 0 or 1 modulo the q's. If B contains 2 consecutive classes $(m) \neq 1$ and $(m+1)$, then $m^2 \equiv m \pmod M$, $(m+1)^2 \equiv (m+1) \pmod M$. By subtracting the first congruence from the second, we get $2m \equiv 0 \pmod M$. Hence $(m, M) = M:2$ and therefore $m \equiv 0 \pmod{M:2}$, $m \equiv 1 \pmod 2$. This happens if and only if one of the q's is equal to 2. If so, $M \equiv 2 \pmod 4$. If therefore B satisfies the conditions established for $E(G)$, then all the q's are > 2. Hence the first proposition holds.

2. Let G belong to $q_1,\ldots,q_t$ and $M = q_1\ldots q_t$, then define the idempotent classes (m_j) modulo M by the congruences $m_j \equiv 1 \pmod{q_j}$, $m_j \equiv 0 \pmod{q_k}$, for $k \neq j$ both indices running over $1,\ldots, t$. The t groups $G^{m_j} = H_j$ are non-commutative normal subgroups of G. From the idempotence of (m_j) follows $m_j^2 \equiv m_j \pmod M$, but for $j \neq k$, $m_j m_k \equiv 0 \pmod M$. If therefore α is an element of H_j, then $\alpha^{m_j} = \alpha$. Let β be a product $\beta = bc\ldots u$ of elements which belong to various groups H_k, where $k \neq j$; as $\{ m_j \}$ is an endomorphism, $\beta^{m_j} = b^{m_j} c^{m_j} \ldots u^{m_j} = 1$. Hence $H_j \cap \prod_{j \neq k} H_k = 1$. The H's admit therefore a direct product $K = H_1 \times H_2 \times \ldots \times H_t$. We shall show now that $K = G$, since it contains every element a of G. As the highest common factor $(m_1, m_2, \ldots, m_t, M) = 1$, the congruence $\Sigma x_j m_j \equiv 1 \pmod M$ has a solution; now $a^{m_j} = a_j \in H_j$ and therefore $a = a_1^{x_1} a_2^{x_2} \ldots a_t^{x_t}$ lies in K. Hence $G = K$. The mapping $\{ q_j \}$ is an endomorphism for H_j since it maps

the group on 1. Suppose $1 < f < q_j$, then $fm_j \equiv f \pmod{q_j}$ and therefore $\{fm_j\}$ cannot be an endomorphism of G. Thus $\{f\}$ is not an endomorphism of H_j and therefore $\{m\}$ is an endomorphism of H_j if and only if $m \equiv 0 \pmod{q_j}$. Hence the second proposition follows.

3. It has been proved in Note I (p.6.) that $E(G \times K) = E(G) \cap E(K)$. If therefore H_1 belongs to q_1, H_2 to $q_2, \ldots, H_t$ to q_t, then $\{m\}$ is an endomorphism of $G = H_1 \times H_2 \times \ldots \times H_t$ if and only if $m \equiv 0$ or 1 modulo every q. When the q's are relatively prime, then G belongs to $q_1, \ldots, q_t$; when $q_1 = \ldots = q_t = q$, then G belongs to q. To prove the 3rd proposition, it suffices therefore to show that to every $q > 2$, there exists a finite group belonging to it. This statement however is an immediate consequence of the following two lemmas.

LEMMA 1. *If $q = p^n > 2$, where p is a prime number, then there exists a finite group which belongs to q.*

LEMMA 2. *Suppose there exists a finite group belonging to r, let $(r,s) = 1$ and $rs = q$, then there exists a finite group belonging to rs if one of the following conditions holds : (i) $s = 2$, (ii) there exists a finite group belonging to s.*

PROOF OF LEMMA 1. Let a,b,c run over the classes of residues (mod q) and consider the finite groups with the elements $[a,b,c]$ and the law of composition $[a,b,c]$ $[a', b', c'] = [a+a', b+b', c+c'+2ab']$. This definition satisfies the group-axioms. There is $[a,b,c]^m = [ma, mb, mc+m(m-1)ab]$ and a comparison between $[a,b,c]^m$ $[a',b',c']^m$ and $([a,b,c]\ [a',b',c'])^m$ shows that $\{m\}$ is an endomorphism if and only if $m(m-1) \equiv 0 \pmod{q}$. Thus the group belongs to q if p is a prime number and $q = p^n > 2$.

PROOF OF LEMMA 2. Let L be an arbitrary finite group of order $w > 1$ and $\lambda_0 = 1, \lambda_1, \ldots, \lambda_{w-1}$ its elements.

The group L admits of a regular representation by which every element is represented by a permutation

$$\lambda_\nu : \quad \begin{pmatrix} 1, \ldots, w \\ \nu_1, \ldots, \nu_w \end{pmatrix}; \qquad (3)$$

thus for $\nu > 0$ and arbitrary j, $\nu_j \neq j$. Suppose C is a finite group which belongs to r; we construct w groups $C_1, \ldots, C_w$ which are isomorphic to C and we denote their elements with $c_{i,k}$ in such a manner that the second index shows the group C_k to which the element belongs and, the first index remaining unaltered, a change of the second index k into j, say, produces an isomorphism $C_{-i,1} \ldots C_{-u,w}$ of C_k and C_j. Let $D = C_1 \times \ldots \times C_w$ and denote its elements with small latin letters a, $b, \ldots$; every permutation of the second indices generates an automorphism of D. In particular the permutations (3) form a group of automorphisms which is isomorphic to L. D belongs to r. Extend now D by L with the law of composition $C_{i,k} \lambda_\nu = \lambda_\nu C_{i,\nu_k}$ to a group G the elements of which will be denoted by small Greek letters $\alpha, \beta, \ldots$ L is a subgroup and D a normal subgroup of G. Every element of G can be represented in one and only one manner by $\alpha = \lambda_\nu a$ and there is $\lambda_\nu a \lambda_\mu b = \lambda_\nu \lambda_\mu d$. If $\{s\}$ maps L on 1, then $\alpha^s = k$ and therefore $\alpha^{rs} = 1$. Hence rs is divisible by $M(G)$; on the other hand r is the smallest exponent mapping the subgroup D of G on an Abelian group and therefore $M(G)$ is divisible by r. Suppose $G = D \times T$, then T contains an element $\lambda_\nu a$ with $\nu \neq 0$ which must be commutative with every element of D, e.g. with $c_{k,1} \neq 1$. But $c_{k,1} \lambda_\nu a = \lambda_\nu c_{k,\nu_1} a$ which is different from $\lambda_\nu a c_{k,1}$ since the components in C_1 of $c_{k,\nu_1} a$ and $a c_{k,1}$ are different. Hence D cannot be a direct factor of G. To prove the lemma, we specialize L in two different ways: (*i*) $L = 1$, λ is of order 2, $s = 2$, $r = 2m+1$. $c_{k,1} \lambda = \lambda c_{k,2}$; hence $(c_k, \lambda)^r = \lambda c_{k,1}{}^m c_{k,2}{}^{m+1} \neq \lambda = \lambda^r c_{k,1}{}^r$. Thus $\{r\}$ is not endomorphic for G and therefore $M(G) = 2r$, $A(G) = 1$. If $\{m\}$ is endomorphic for G, then $m \equiv 0$ or

1 (mod r) and therefore $\equiv 0, 1, r, r+1 \pmod{2r}$; but if $\{r+1\}$ is endomorphic, then also $\{1-(r+1)\} \equiv \{r\}$ (mod $2r$), and this has already been proved to be impossible. Hence G belongs to $2r$.

(*ii*) L belongs to s, where $(r, s) = 1$. If $\{m\}$ maps L and D on Abelian groups, then $m \equiv 0 \pmod{rs}$; hence $M(G) = rs$, $A(G) = 1$. If $\{m\}$ is endomorphic, then $m \equiv 0$, 1 (mod r) and (mod s). Thus if G does not belong to rs, it must belong to r, s. This however implies that G^m is a direct factor when $m \equiv 0 \pmod{s}$, $m \equiv 1 \pmod{r}$; for the first congruence, G^m is contained in D, for the second congruence, $D^m = D$; hence $G^m = D$. As it has been proved that D cannot be a direct factor of G, the group G belongs to rs.

This finishes the proof of the lemma and of the third proposition.

ON INTEGRAL REPRESENTATION FOR WHITTAKER FUNCTIONS.

BY

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1. Since $\phi(p) \doteqdot f(x)$ and $\phi(p+\alpha)\, p/(p+\alpha) \doteqdot e^{-\alpha x} f(x)$,
$\alpha > 0$, (1.1)

when $\phi(p) = p \int_0^\infty e^{-px} f(x)\, dx$ provided the integral converges, and since(1)

$$W_{k,m}(p) e^{p/2} p^{\frac{1}{2}-m} = \frac{p}{\Gamma(\frac{1}{2}-k+m)} \int_0^\infty e^{-px} x^{-\frac{1}{2}-k+m} (1+x)^{m+k-\frac{1}{2}}\, dx$$

$R(\frac{1}{2}-k+m) > 0$, (1.2)

we have

$$e^{p/2} W_{k,m}(p)\, p^{\frac{1}{2}-m} \doteqdot \frac{x^{-\frac{1}{2}-k+m}(1+x)^{m+k-\frac{1}{2}}}{\Gamma(\frac{1}{2}-k+m)} \qquad (1.3)$$

and

$$p\, e^{\frac{p+a}{2}} (p+a)^{-\mu-\frac{1}{2}} W_{\lambda,\mu}(p+a)$$
$$\doteqdot \frac{e^{-ax} x^{-\frac{1}{2}-\lambda+\mu} (1+x)^{-\frac{1}{2}+\lambda+\mu}}{\Gamma(\frac{1}{2}-\lambda+\mu)}, \; R(\tfrac{1}{2}-\lambda+\mu) > 0, \; a > 0. \quad (1.4)$$

Again since

$$p^{-a}(1+p)^{a-c} = \frac{1}{\Gamma(c)} \int_0^\infty e^{-x(p+1)} x^{c-1} {}_1F_1[a;\, c;\, x]\, dx,$$
$$R(c) > 0, \; p > 0, \quad (1.5)$$

(a result that can be verified by term by term integration, which is quite legitimate) we have for suitable values of a and c,

$$p^{\lambda-k-m-\mu+1} (1+p)^{k-\lambda+m-\mu} \doteqdot \frac{e^{-x} x^{2\mu-2m-1}}{\Gamma(2\mu-2m)}$$
$$\times {}_1F_1[k-\lambda+\mu-m;\, 2\mu-2m;\, x], \; R(\mu-m) > 0. \quad (1.6)$$

Hence using (1.4) and (1.6) in Parseval's theorem, we have, after a little rearrangement,

$$W_{k,m}(z)$$
$$= \frac{\Gamma(\frac{1}{2}-\lambda+\mu)\, z^{m-\mu}}{\Gamma(2\mu-2m)\, \Gamma(\frac{1}{2}-k+m)} \int_0^\infty e^{-t/2} t^{2\mu-2m+1} \left(1+\frac{t}{z}\right)^{-\mu-\frac{1}{2}}$$
$$\times W_{\lambda,\mu}(z+t)\; {}_1F_1[k-\lambda+\mu-m;\, 2\mu-2m;\, t]\, dt,$$
$$R(\mu-m) > 0, \; R(\tfrac{1}{2}-\lambda+\mu) > 0, \; R(\tfrac{1}{2}-k+m) > 0, \; R(z) > 0. \quad (1.7)$$

As a particular case, when $k-m = \mu-\lambda$, we have[2]

$$W_{k,m}(z) = \frac{z^{m-\mu}}{\Gamma(2\mu-2m)} \int_0^\infty e^{-t/2} t^{2\mu-2m-1} \left(1+\frac{t}{z}\right)^{-\mu-\frac{1}{2}}$$
$$\times W_{\lambda,\mu}(z+t)\; dt, \quad R(\mu-m) > 0, \quad R(\tfrac{1}{2}-k+m) > 0, \quad z \neq 0$$
$$|\arg z| < \pi; \quad (1.8)$$

while for $k+m = \mu+\lambda$, we obtain [3]

$$W_{k,m}(z) = \frac{\Gamma(\frac{1}{2}-\lambda+\mu)\, z^{m-\mu}}{\Gamma(2\mu-2m)\Gamma(\frac{1}{2}-k-m)} \int_0^\infty e^{t/2} t^{2\mu-2m-1}$$
$$\times (1+t/z)^{-\mu-\frac{1}{2}} W_{\lambda,\mu}(z+t)\; dt$$

valid under the same conditions of (1.7). (1.9)

Again taking $\lambda = -\mu+\frac{1}{2}$ and noting that

$$W_{-\mu+\frac{1}{2},\mu}(z) = e^{-z/2} z^{-\mu+\frac{1}{2}}$$

we get [4] from (1.7)

$$W_{k,m}(z) = \frac{e^{-z/2} z^{m-2\mu+\frac{1}{2}} \Gamma(2\mu)}{\Gamma(2\mu-2m)\,\Gamma(\frac{1}{2}-k+m)} \int_0^\infty e^{-t} t^{2\mu-2m-1}$$

$$\times (1+t/z)^{-2\mu} {}_1F_1\left[\begin{matrix} k+2\mu-m-\frac{1}{2}; \\ 2\mu-2m;\ t \end{matrix}\right] dt,\ R(\mu-m)>0,\ R(\mu)>0,$$

$R(\frac{1}{2}-k+m)>0$, $|\arg x|<\pi$ and $z\neq 0$. (1.10)

Several particular cases of (1.7) that can be deduced from the relations between Kummer's function and other functions are left to the reader.

2. Now starting with the integral(5)

$${}_2F_1[\alpha, \beta; \gamma, z] \doteq \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-t} t^{\lambda-1} \cdot {}_2F_2[\alpha, \beta; \gamma, \lambda;\ tz]\,dt \quad (2.1)$$

we get after transformation,

$$p\,{}_2F_1\left[\begin{matrix} l+m+\frac{1}{2},\ l-m+\frac{1}{2}; \\ l-k+1,\ -p \end{matrix}\right]$$

$$= \frac{1}{\Gamma(\nu+1)} \int_0^\infty e^{-t/p} (t/p)^\nu = {}_2F_2\left[\begin{matrix} l+m+\frac{1}{2}, l-m+\frac{1}{2}; -t \\ \nu+1,\ l-k+1, \end{matrix}\right] dt; \quad (2.2)$$

and because(6)

$$e^{-\frac{1}{2}t}\, t^{l-1}\, W_{k,m}(t)$$

$$\doteq \frac{\Gamma(l+m+\frac{1}{2})\,\Gamma(l-m+\frac{1}{2})}{\Gamma(l-k+1)}\, p\,{}_2F_1\left[\begin{matrix} l+m+\frac{1}{2}, l-m+\frac{1}{2} \\ l-k+1, -p \end{matrix}\right], \quad (2.3)$$

$$R(l+m+\tfrac{1}{2})>0 \quad (2.4)$$

and $\quad (xt)^{\nu/2} J_\nu(2\sqrt{xt}) \doteq e^{-t/p} (t/p)^\nu$

we have on taking the originals of both the sides in (2.2) with the help of (2.3) and (2.4)

$$e^{-\frac{1}{2}x} x^{l-1} W_{k,m}(x)$$

$$= \frac{\Gamma(l+m+\frac{1}{2})\,\Gamma(l-m+\frac{1}{2})}{\Gamma(l-k+1)\,\Gamma(\nu+1)}\, x^{\nu/2}$$

$$\times \int_0^\infty t^{\nu/2} J_\nu(2\sqrt{xt})\, {}_2F_2\left[\begin{matrix} l+m+\frac{1}{2},\ l-m+\frac{1}{2} \\ \nu+1,\ l-k+1,\ -t \end{matrix}\right] dt \quad (2.5)$$

$$R(\nu+1)>0,\ R(l\pm m+\tfrac{1}{2})>0,\ R(l\pm m-\tfrac{1}{4}-\nu/2)>0.$$

or

$$e^{-\frac{1}{4}x^2} x^{2l-\nu-2} W_{k,m}(x^2/2)$$

$$= \frac{\Gamma(l+m+\frac{1}{2})\,\Gamma(l-m+\frac{1}{2})}{\Gamma(l-k+1)\,\Gamma(\nu+1)}\, 2^{l-\nu-1}$$

$$\times \int_0^\infty t^{\nu+1} J_\nu(xt)\, {}_2F_2\left[\begin{matrix} l+m+\frac{1}{2}, l-m+\frac{1}{2}, \\ \nu+1,\ l-k+1, -t^2/2 \end{matrix}\right] dt. \quad (2.6)$$

$R(\nu+1) > 0$, $R(l\pm m+\frac{1}{2}) > 0$, $R(l\pm m-\frac{1}{2}\nu-\frac{1}{4}) > 0$.

The result (2.6) together with [7]

$$\int_0^\infty e^{-\frac{1}{4}y^2} y^{2l-\nu-1} J_\nu(xy)\, W_{k,m}(y^2/2)\, dy$$

$$= \frac{2^{l-\nu-1}x^\nu\,\Gamma(l+m+\frac{1}{2})\,\Gamma(l-m+\frac{1}{2})}{\Gamma(\nu+1)\,\Gamma(l-k+1)}\; {}_2F_2\left[\begin{matrix} l+m+\frac{1}{2},\; l-m+\frac{1}{2}; \\ \nu+1,\; l-k+1;\; -\frac{1}{2}x^2 \end{matrix}\right],$$

$$R(l\pm m+\tfrac{1}{2}) > 0 \qquad (2.7)$$

gives rise to the following

THEOREM: *If* $\phi(x) = e^{-\frac{1}{4}x^2}x^{2l-\nu-\frac{3}{2}}W_{k,m}(x^2/2)$

and

$$\Psi(x) = \frac{2^{l-\nu-1}x^{\nu+\frac{1}{2}}\,\Gamma(l+m+\frac{1}{2})\,\Gamma(l-m+\frac{1}{2})}{\Gamma(\nu+1).\Gamma(l-k+1)}$$

$$\times {}_2F_2\left[\begin{matrix} l+m+\frac{1}{2},\; l-m+\frac{1}{2}, \\ \nu+1,\; l-k+1\;;\; -\frac{1}{2}x^2 \end{matrix}\right],$$

then $\phi(x)$ *and* $\Psi(x)$ *are Hankel-transforms of each other of order* ν, *if*

$R(\nu+1) > 0$, $R(l\pm m+\frac{1}{2}) > 0$ *and* $R(l\pm m-\frac{1}{2}\nu-\frac{1}{4}) > 0$. (2.8)

In particular when $l=\nu-m+\frac{1}{2}$, and $k=\nu-m-2\mu+\frac{1}{2}$, i. e. when the generalized hypergeometric function reduces to Kummer's function, we have from (2.6)

$$e^{-\frac{1}{4}x^2}x^{\nu-2m-1}W_{\nu-m-2\mu+\frac{1}{2},m}(\tfrac{1}{2}x^2)$$

$$= \frac{\Gamma(\nu-2m+1)}{\Gamma(2\mu+1)2^{m+\frac{1}{2}}}\int_0^\infty t^{\nu+1}J_\nu(xt)\;{}_1F_1[\nu-2m+1\;;\;2\mu+1;\;-\tfrac{1}{2}t^2]\,dt,$$

$$R(\nu+1) > 0;\; R(\nu-4m+\tfrac{1}{2}) > 0. \qquad (2.9)$$

This again in consequence of the relations

$${}_1F_1[a\,;\,b\,;\,-z] = e^{-z}\;{}_1F_1[b-a;\,b,\,z]$$

and $\qquad M_{k,m}(x) = e^{-x/2}x^{m+\frac{1}{2}}\;{}_1F_1[\frac{1}{2}-k+m;\;2m+1;\;x]$

can be put in the form [8]

$$e^{-\frac{1}{4}x^2}x^{\nu-2m-1}W_{\nu-m-2\mu+\frac{1}{2},m}(\tfrac{1}{2}x^2)$$

$$= \frac{\Gamma(\nu-2m+1)}{\Gamma(2\mu+1)}\,2^{\mu-m}\int_0^\infty e^{-\frac{1}{4}t^2}t^{\nu-2\mu}J_\nu(xt)\,M_{\nu-\mu-2m+\frac{1}{2},\mu}(\tfrac{1}{2}t^2)\;dt,$$

$$R(\nu+1) > 0,\; R(\nu-4m+\tfrac{1}{2}) > 0, \qquad (2.10)$$

showing that the functions

$$e^{-\frac{1}{4}x^2}x^{\nu-2m-\frac{1}{2}}W_{\nu-m-2\mu+\frac{1}{2},m}(\tfrac{1}{2}x^2)$$

and

$$\frac{\Gamma(\nu-2m+1)\, 2^{\mu-m}}{\Gamma(2\mu+1)} e^{-\frac{1}{4}x^2} x^{\nu-2\mu-\frac{1}{2}} M_{\nu-\mu-2m-\frac{1}{2},\,\mu}(\tfrac{1}{2}x^2)$$

are Hankel-transforms of each other, when

$R(\nu+1)>0$, $R(\nu-4m+\frac{1}{2})>0$ and $R(\nu-2m+1)>0$.

Various other known results involving Weber's parabolic cylinder functions and Laguerre polynomials, etc. can similarly be shown to be special cases of (2.8).

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ON THE HARMONIC TRANSVERSALS OF TWO PAIRS OF QUADRICS AND A STRAIGHT LINE.

BY

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1. In two previous papers*, the author has discussed the locus of the harmonic transversals of two given quadrics that also intersect two given curves. The object of the present paper is to discuss the locus of the harmonic transversals of two pairs of quadrics that intersect also a given straight line. The locus obtained is a ruled surface of order 8, which has the given line as a multiple line of order 4.

2. The harmonic transversals through a given point P of a pair of quadrics generate a quadric cone with P as vertex. The equation of this cone contains the coordinates of P in the second degree. Given two pairs of quadrics,

* Rangachariar and N. Chatterji, *Bull. Calcutta Math. Soc.* 35 (1943), V. Rangachariar, *Bull. Calcutta Math. Soc.* 36 (1944).

we get correspondingly two cones having P as their common vertex. These two cones intersect in general in 4 straight lines through P and therefore *through any point in space four straight lines can be drawn to intersect two pairs of quadrics in pencils in harmonic ranges.* If P lies on a straight line σ, the coordinates of P are linear functions of a parameter t and the t-eliminant between the equations of the two cones is the locus of the harmonic transversals. Clearly this locus is a scroll of order 8. If P lies on a curve of order m, the corresponding locus is of order $4m$. *Hence the locus of harmonic transversals of two pairs of quadrics which also intersect a given straight line is a scroll* Σ *of order* 8.

3. Through every point P of σ, four generators of Σ will pass and hence *the line* σ *will be a multiple line of order* 4 *on* Σ. For certain positions of P however, two of these four lines will coincide and these positions of P will be pinch points on σ. Taking tetrahedral coordinates, the x, w eliminant between the equations of the two cones is an equation of the fourth degree in y/z in which the coefficients are certain functions of the 8th degree in $\alpha, \beta, \gamma, \delta$ —the tetrahedral coordinates of P. The condition that this equation in y/z has two equal roots is the vanishing of its discriminant which is of degree 48 in $\alpha, \beta, \gamma, \delta$. Hence two of the four generators through P coincide only when P lies on a certain surface of order 48. This meets σ in 48 points. Hence *there are* 48 *pinch points of* Σ *lying on* σ.

4. Any plane π through σ interests Σ in the straight line σ counted 4 times and a certain quartic curve. This plane π however meets the two pairs of quadrics in two pairs of conics. The four common tangents to the two ϕ-conics of the two pairs of conics are evidently generators of Σ and therefore form the quartic curve mentioned above. *Hence any plane through* σ *meets* Σ *in four straight lines other than* σ. These four lines in π intersect one another in 6 points which trace out a certain curve τ of order 6 as the

plane π is varied. *This curve τ is a double curve on Σ because through every point of τ two distinct generators of Σ pass.*

5. *The surface Σ has no other double point.* For if P be a double point which does not lie on σ or τ, the plane π through σ and P meets Σ in four generators of Σ meeting in 9 points no one of which coincides with P. But if P be a double point on Σ, it is also a double point of the section of Σ by π. But this is impossible unless P coincides with one of the 6 points.

6. Any plane π of general position intersects Σ in a curve ρ of order 8 which has a 4-ple point at the point O where σ meets π. The tangents to ρ at O are the intersections with π of the four planes one through each of the four generators through O and σ. π meets τ in six points which will be double points on ρ, the tangents at the double points being the intersections with π of the two planes one through each of the two generators of Σ and the tangent to τ at the point. The curve ρ can have no other double point. For if P be such a double point, the generator through P of σ must lie in π (because π is the tangent plane to Σ at P) and must pass through O. Hence the curve ρ can have no other double point if π does not contain a generator of Σ.

If however π passes through one generator d through O in σ, π meets Σ in a residual curve ρ' of order 7. Any line through O in π meets ρ' in 4 points other than O, these being the points where the four generators in the plane of σ and this line, meets it. O is thus a 4-ple point on ρ' (as is also otherwise evident). The generator d however can meet ρ' in three distinct points (other than O) only, these being the points of intersection with d of the three other generators of Σ coplanar with d. Hence d touches ρ' at a point. *Hence a plane of general position meets Σ in a curve of order* 8 *having one 4-ple point and* 6 *other double points. If however the plane contains a generator of Σ,*

it meets Σ *in a residual curve of order* 7 *having a triple point and touching d at a point. It has three other double points.*

7. *The surface* Σ *can also be generated by trisectants of* τ *that intersect* σ. To generate the surface thus, take any plane through σ meeting τ in 6 points which are the vertices of a complete quadrilateral whose sides are the generators of Σ.

8. The section of Σ by a plane of general position is, as shown above, a curve of order 8 having 6 isolated double points and one 4-ple point and therefore of deficiency 9. *Hence the genus* p *of* Σ *is* 9. *The number of torsal generators* (as also pinch points) *is* $2(n+2p-2)=48$ which agrees with the number of pinch-points found above. The characteristics of the surface can now be easily found.

9. We have seen above that a plane π through σ meets Σ, in general, in 4 straight lines. If however π touches S_1 (one of the quadrics) at P, π cuts S_1 in a pair of straight lines. The ϕ-conic of the conic section γ of π and S_2 and this line-pair is a pair of straight lines through P. The common tangents to this line-pair and the ϕ-conic of the two conics of section by π of the second pair of quadrics is again a line-pair through P. These two lines are torsal generators of Σ. We have thus 16 torsal generators lying in 8 planes through σ. The 8 points P corresponding to the 8 planes are sextuple points on τ. *Hence* 8 *pairs of torsal generators are coplanar lying in* 8 *planes through* σ. No other pair of torsal generators can be coplanar. If d be one of the 32 other torsal generators, the plane π through σ and d can contain two other generators only, for d is a double generator. Hence the two ϕ conics in π must touch at a point of d. The points where the torsal generator d meets the other two generators in π are double points of τ. *Hence* τ *has* 8 *sextuple points and* 64 *double points.*

10. The quartic curve common to S_1 and S_2—one of the two pairs of quadrics—cuts Σ in 32 points. If A be one of these points, the generator of Σ through A cuts S_1, S_2 in points A, B; A, D (say) which form a harmonic range. Hence B or D must coincide with A. *Hence the generator of Σ through A must touch either S_1 or S_2 at A.* Conversely a generator of Σ which touches either S_1 (or S_2) at a point A must cut S_2 (or S_1) at the same point. Hence *there are* 16 *generators of Σ which touch S_1 at points of the common quartic curve.*

11. Lines through a given point cutting two sets of three quadrics each in involution generate two cubic cones having 9 common generators. The equations of these two cones contain the coordinates of the point in the third degree. Hence if the point lies on a straight line, the transversals generate a surface of order 18 having the line σ as a multiple line of order 9. Any plane through σ cuts the quadrics in two sets of three conics whose cubic envelopes have 9 common tangents. These 9 common tangents together with σ from the complete intersection of the surface by the plane. The 9 common tangents intersect in pairs in 36 points which trace out a curve τ of order 36. The surface has properties similar to those of Σs We propose to investigate the properties of this surface in a subsequent paper.

A NOTE ON THE LOWER ORDER OF INTEGRAL FUNCTIONS.

BY

S. M. SHAH, *Aligarh*.

[Received 29 December, 1944.]

Let $f(z) = \sum_0^\infty a_n z^n$ be an integral function and let

$$\theta(n) = \left\{ \frac{n \log n}{\log (1/|a_n|)} \right\}.$$

A necessary and sufficient condition that $\Sigma a_n z^n$ be an integral function of finite order ρ is that*

$$\rho = \limsup_{n\to\infty} \theta(n). \tag{1}$$

A similar result for the lower order λ, viz.

$$\lambda = \liminf_{r\to\infty} \frac{\log \log M(r)}{\log r} = \liminf_{n\to\infty} \theta(n) \tag{2}$$

does not always hold. In fact for the series,

$$e^{z^2}+e^z = 2+z+z^2(1/1!+1/2!)+\ldots,$$

$\liminf \theta(n) = 1$; $\lambda = \rho = 2$.

We prove here

THEOREM 1. *If* $f(z) = \Sigma a_n z^n$ *is an integral function of lower order* λ $(0 \leqslant \lambda < \infty)$, *then*

$$\lambda \geqslant \liminf_{n\to\infty} \frac{n \log n}{\log 1/|a_n|}. \tag{3}$$

COR. *If* $\lim_{n\to\infty} \frac{n \log n}{\log 1/|a_n|} = l$, *where* $0 \leqslant l \leqslant \infty$, *then* $f(z) = \Sigma a_n z^n$ *is an integral function of regular growth*† *and of order* l.

The example given above shows that $f(z)$ *may be of regular growth and* $\lim \theta(n)$ *may not exist.* When $\lim_{n\to\infty} \theta(n) = \infty$, $f(z)$ may not be an integral function. However *if for an integral function* $f(z)$, $\lim \theta(n) = \infty$ *then*, we have from Theorem 1, that $f(z)$ *is of infinite order for which* $\lambda = \rho = \infty$.

THEOREM 2. *If* $f(z) = \Sigma a_n z^n$ *is an integral function of lower order* λ $(0 \leqslant \lambda \leqslant \infty)$ *such that* $|a_n/a_{n+1}|$ *is a non-decreasing function of* n *for* $n \geqslant n_0$, *then*

$$\lambda = \liminf_{n\to\infty} \frac{n \log n}{\log 1/|a_n|}. \tag{4}$$

We note that the hypothesis of Theorem 2 does not imply that $f(z)$ is of regular growth. In fact we have

THEOREM 3. *There exists an integral function* $f(z) = \Sigma a_n z^n$ *for which*

* Titchmarsh : *Theory of Functions*, p. 253.

† Valiron : *Integral Function*, pp. 41-44.

(i) $a_n \geqslant 0$

(ii) $\{a_n/a_{n+1}\}$ *is a steadily increasing function of* n

and. (iii) $\rho > \lambda$.

PROOF OF THEOREM 1. Let $\liminf \theta(n) = \mu$. Since $f(z)$ is an integral function, $\mu \geqslant 0$. If $\mu = 0$ then (3) is obvious. Suppose $0 < \mu < \infty$; then we have

$$|a_n| > n^{-\frac{n}{\mu-\varepsilon}} \text{ for all } n \geqslant N.$$

Let $r_n = 2\, n^{\frac{1}{\mu-\varepsilon}}$. If $r_n \leqslant r \leqslant r_{n+1}$ $(n > N)$ then $M(r) \geqslant |a_n| r^n$ $\geqslant |a_n| r_n^n > n^{-\frac{n}{\mu-\varepsilon}} e^{n \log r_n} = e^{n \log 2}$. Hence

$$\log M(r) \geqslant n \log 2 = \log 2 \,.\{(r/2)^{\mu-\varepsilon} - 1\}$$

for all large r. Hence $\lambda \geqslant \mu$.

If $\mu = \infty$ the above argument shows that $\lambda = \infty$.

The corollary follows from (1) and (3).

PROOF OF THEOREM 2. Let $\mu(r)$ denote the maximum term, $\nu(r)$ its rank and $\phi(n) = |a_n/a_{n+1}|$. Then $\phi(n) > \phi(n-1)$ for an infinity of n; for if otherwise $\phi(n) = \phi(n+1) = \phi(n+2) = \ldots$ad inf for $n > p$ say, and hence the radius of convergence of the series $\Sigma a_n z^n$ would be finite. When $\phi(n) > \phi(n-1)$ the term $a_n z^n$ becomes a maximum term and we have $\mu(r) = |a_n| r^n$, $\nu(r) = n$ for $\phi(n-1) \leqslant r < \phi(n)$. Now*,

$$\lambda = \liminf_{r\to\infty} \frac{\log \nu(r)}{\log r}.$$

Suppose first that $0 < \lambda < \infty$. Then

$$\nu(r) > r^{\lambda-\varepsilon} \text{ for } r > R = R(\varepsilon). \tag{5}$$

We shall suppose that $|z| = r > R$. Let $a_{m_1} z^{m_1}$ and $a^{m_2} z^{m_2}$ $(m_1 > n_0)$ be two consecutive maximum terms so that $m_1 \leqslant m_2 - 1$ and let $m_1 < n \leqslant m_2$. Since $a_{m_1} z^{m_1}$ is a maximum term, we have for

$$R < \phi(m_1 - 1) \leqslant r < \phi(m_1),\ \nu(r) = m_1.$$

Hence for every r in this interval

$$m_1 = \nu(r) > r^{\lambda-\varepsilon}$$

* J. M. Whittaker: *Jour. Lond. Math. Soc.* 8 (1933), 20-7.

In particular $m_1 > \{\phi(m_1) - a\}^{\lambda-\varepsilon}$, where

$$R = \text{Min}\left\{1, \frac{\phi(m_1) - \phi(m_1 - 1)}{2}\right\}.$$

Further we have*

$$\phi(m_1) = \phi(m_1 + 1) = \ldots = \phi(n - 1).$$

Hence

$$\phi(n_0)\ \phi(n_0 + 1) \ldots \phi(n - 1) = |a_{n_0}/a_n|$$
$$\leqslant |a_{m_1 - 1}/a_{m_1}|^{m_1 - n_0}\ |a_{m_1}/a_{m_1 + 1}|^{n - m_1}$$
$$< m_1^{\frac{m_1 - n_0}{\lambda - \varepsilon}} \left\{a + m_1^{\frac{1}{\lambda - \varepsilon}}\right\}^{n - m_1}$$
$$< n^{\frac{m_1 - n_0}{\lambda - \varepsilon}}\ 2^{n - m_1}\ n^{\frac{n - m_1}{\lambda - \varepsilon}}.$$

Hence

$$\liminf_{n \to \infty} \theta(n) \geqslant \lambda, \tag{6}$$

which holds also when $\lambda = 0$. If $\lambda = \infty$, then, we write H, where H is any arbitrary large number, instead of $\lambda - \varepsilon$ in (5) and obtain that $\liminf \theta(n) = \infty$. Hence from Theorem 1, we obtain (4).

PROOF OF THEOREM 3. Let $n_1 = 2$, $n_{s+1} = n_s^4$, $s = 1$, 2, 3,..

$$r_1 = 1$$
$$r_m = m \text{ for } n_s \leqslant m < n_s^2$$
$$= n_{s+1} - \frac{n_{s+1} - m}{\{(n_{s+1})!\}^{(n_{s+1})}} \text{ for } n_s^2 \leqslant m \leqslant n_{s+1}$$

and $s = 1, 2, 3, \ldots$ Let

$$f(z) = 1 + \sum_{1}^{\infty} \frac{z^n}{r_1 r_2 \ldots r_n}.$$

Then $a_n = \dfrac{1}{r_1 r_2 \ldots r_n}$ is positive and decreasing. Also

$\dfrac{a_n}{a_{n+1}} = r_{n+1}$ which is a steadily increasing function of n.
Also

* When $m_1 = m_2 - 1$, the product $\Pi\phi(k)$, $(m_1 \leqslant k \leqslant n - 1)$ will consist of one term only.

$$\theta(n) = \frac{n \log n}{\log r_1 + \ldots + \log r_n}$$

Hence

$$\theta(n_{s+1}) \sim \frac{4 n_s^4 \log n_s}{(n_s^4 - n_s^2) \log (n_s^4)} \sim 1.$$

$$\theta([n_s^2 \log n_s]) \sim \frac{n_s^2 \log n_s \log [n_s^2 \log n_s]}{(n_s^2 \log n_s - n_s^2) \log (n_s^4) + O(n_s^2 \log n_s)}$$

$$\sim \tfrac{1}{2}.$$

It is easily seen that $\limsup_{n\to\infty} \theta(n) = 1$; $\liminf_{n\to\infty} \theta(n) = \frac{1}{2}$. Hence $f(z)$ is an integral function of order 1 and lower order $\frac{1}{2}$.

THE MAXIMUM TERM OF AN ENTIRE SERIES (II)

BY

S. M. SHAH, *Aligarh.*

[Received 4th December, 1944.]

Let $\mu(r)$ be the maximum term for $|z| = r$ of the entire series $f(z) = \sum_0^\infty a_n z^n$ and $\nu(r)$ its rank. In a previous note* I have proved a relation between $\mu(r)$ and $\nu(r)$. I prove here†

THEOREM. *If* $0 < r_1 < r_2$ *then*

$$\left(\frac{r_2}{r_1}\right)^{\nu(r_1)} \leqslant \frac{\mu(r_2)}{\mu(r_1)} < \left(\frac{r_2}{r_1}\right)^{\nu(r_2)} \qquad (1)$$

PROOF. Let

$$\mu(r_1) = |a_n| r_1^n ; \; n = \nu(r_1)$$
$$\mu(r_2) = |a_m| r_2^m ; \; m = \nu(r_2).$$

Since $r_1 < r_2$, $m \geqslant n$. Now

$$\mu(r_2) = (r_2/r_1)^m |a_m| r_1^m \leqslant (r_2/r_1)^m \mu(r_1).$$

* *Math. Student*, 10 (1942), 80-2.

† *Cf.* Polya und Szego: *Aufgaben und Lehrsatze* II P. 3-4. Questions 17 and 23.

Also $\mu(r_2) \geqslant |a_n| r_2^n = |a_n| r_1^n (r_2/r_1)^n = \mu(r_1) (r_2/r_1)^n$.
Hence the result follows.

COR. *If α be a constant, $0 < \alpha < 1$, then**

$$\lim_{r\to\infty} \frac{\mu(\alpha r)}{\mu(r)} = 0. \tag{2}$$

For from (1) we have, putting $r_1 = \alpha r, r_2 = r$

$$\alpha^{\nu(\alpha r)} \geqslant \frac{\mu(\alpha r)}{\mu(r)} \geqslant \alpha^{\nu(r)} \tag{3}$$

As $r \to \infty$, $\nu(\alpha r)$ and $\nu(r)$ tend to infinity and the extreme terms in (3) tend to zero. Hence (2) follows.

* We suppose there that $f(z)$ does not reduce to a polynomial.

CONGRUENCE PROPERTIES OF RAMANUJAN'S FUNCTION $\tau(n)$ (II)

BY

K. G. RAMANATHAN, *University of Madras.*

[Received 26th October, 1944.]

1. Let $\tau(n)$ be Ramanujan's function defined by

$$\sum_{n=1}^{\infty} \tau(n)\, x^n = x\left[(1-x)(1-x^2)\ldots\right]^{24}.$$

It was recently proved by me* that

$$\tau(kn-1) \equiv 0 \pmod{k}, \tag{1.1}$$

where $k = 3, 4, 6, 8, 12$ or 24 and $n \geqslant 1$. The object of this paper is to find the residues of $\tau(n)$ to the moduli 3, 4 and 7. The main theorems are :

THEOREM A. *Let $M = 3^\alpha \prod_p p^\beta \prod_q q^\gamma$, where $p \equiv -1 \pmod 3$ and $q \equiv 1 \pmod 3$*†. *Then*

$$\tau(M) \equiv 0 \pmod 3 \tag{1.2}$$

if $\alpha > 0$ or any β is odd. When none of these conditions is satisfied

$$\tau(M) \equiv \prod_\gamma (1+\gamma) \pmod 3. \tag{1.3}$$

* *Proc. Indian Academy of Sciences*, 19 (1944), 146-8.

† p and q stand for rational prime numbers.

THEOREM B. *Let* $M = 2^{\alpha}\prod_{p} p^{\beta} \prod_{q} q^{\gamma}$, *where* $p \equiv -1 \pmod 4$ *and* $q \equiv 1 \pmod 4$. *Then*

$$\tau(M) \equiv 0 \pmod 4 \qquad (1.4)$$

if $\alpha > 0$ *or any* β *is odd. When none of these conditions is satisfied*

$$\tau(M) \equiv \prod_{\gamma}(1+\gamma) \pmod 4. \qquad (1.5)$$

THEOREM C. *Let* $M = 7^{\alpha}\prod_{p} p^{\beta} \prod_{q} q^{\gamma}$, *where* p *is a quadratic non-residue of* 7 *and* q *a quadratic residue of* 7. *Then*

$$\tau(M) \equiv 0 \pmod 7 \qquad (1.6)$$

if $\alpha > 0$ *or any* β *is odd. When none of these conditions is satisfied*

$$\tau(M) \equiv M \prod_{\gamma}(1+\gamma) \pmod 7. \qquad (1.7)$$

Theorem C was also stated and proved by J. R. Wilton* who used ideal theory in quadratic fields. In this paper use is made of results in the theory of elliptic modular functions which Ramanujan has proved by elementary methods.

2. Ramanujan conjected and Mordell† proved that

$$\tau(m)\tau(n) = \tau(mn), \ (m, n) = 1 \qquad (2.1)$$

$$\sum_{n=1}^{\infty} \tau(n) n^{-s} = \prod_{p} \left[1 - \tau(p)p^{-s} + p^{11-2s}\right]^{-1} \qquad (2.2)$$

the infinite product extending over all the prime numbers 2, 3, 5, ...It is obvious that if k is an integer $\geqslant 2$ then

$$\tau(p^k) = \tau(p)\tau(p^{k-1}) - p^{11}\tau(p^{k-2}). \qquad (2.3)$$

3. PROOF OF THEOREM A.

LEMMA 1. *If* $p \equiv -1 \pmod 3$ *then* $\tau(p^k) \equiv \frac{1}{2}[1+(-1)^k] \pmod 3$.

PROOF. If r is a positive integer

$$\tau(p^{2r}) = \tau(p)\ \tau(p^{2r-1}) - p^{11}\tau(p^{2r-2})$$
$$\equiv \tau(p^{2r-2}) \equiv \ldots \equiv \tau(p^2) \equiv 1 \pmod 3.$$

* *Proc. Lond. Math. Soc.* 31 (1930), 1-10.

† *Proc. Camb. Phil. Soc.* (1919), 117-24.

$$\tau(p^{2r+1}) \equiv \tau(p)\ \tau(p^{2r}) - p^{11}\tau(p^{2r-1})$$
$$\equiv \tau(p^{2r-1}) \equiv \ldots \equiv \tau(p) \equiv 0 \pmod 3.$$

LEMMA 2. *If* $q \equiv 1 \pmod 3$, *then* $\tau(q^k) \equiv (1+k) \pmod 3$.

PROOF. It is easy to see that

$$\sum_{n=1}^{\infty} \tau(n)x^n \equiv x[(1-x^3)(1-x^6)\ldots]^8 \pmod 3.$$

Also it is well known* that

$$x[(1-x^3)\ (1-x^6)\ldots]^8 = \sum_{n=1}^{\infty} f(n)x^n, \tag{2.4}$$

where $f(n) = \frac{1}{8}\sum_{\xi} \xi^3$, $\xi \equiv 2 \pmod 3$ and ξ running through all these solutions of $n = \xi^2 + 3\xi\eta + 3\eta^2$. It is evident that $f(q) = 2\ (v^3 - 9vu^2)$, $v \equiv 1 \pmod 3$ and $q = 3u^2 + v^2$ so that when $q \equiv 1 \pmod 3$

$$f(q) \equiv -1 \pmod 3. \tag{2.5}$$

Thus we have $\tau(q) \equiv -1 \pmod 3$.

From (2.2) we have

$$\sum_{n=0}^{\infty} \tau(q^n)\ q^{-tn} = \left[1 - \tau(q)\, q^{-t} + q^{11-2t}\right]^{-1}$$
$$\equiv (1-q^{-t})^{-2} \pmod 3.$$

Combining these with $\tau(3n) \equiv 0 \pmod 3$, we have the theorem.

4. PROOF OF THEOREM B.

LEMMA 3. *If* $p \equiv -1 \pmod 4$, *then* $\tau(p^k) \equiv \frac{1}{2}[1 + (-1)^k] \pmod 4$.

LEMMA 4. *If* $q \equiv 1 \pmod 4$, *then* $\tau(q^k) \equiv (1+k) \pmod 4$.

PROOF. It is easily seen that

$$\sum_{n=1}^{\infty} \tau(n)\ x^n \equiv x[(1-x^4)\ldots]^6 \pmod 4.$$

Also it is known that*

* *See* Klein und Fricke, *Elliptische Modulfunktionen* Bd. 2; or Mordell, *loc. cit.*

$$x[(1-x^4)(1-x^8)\ldots]^6 = -\tfrac{1}{2}\Sigma\Sigma(\xi^2-\eta^2)\,x^{\xi^2+\eta^2} \qquad (2.6)$$

$$= \sum_{n=1}^{\infty} \phi(n)x^n,$$

where $\phi(n) = -\frac{1}{2}\Sigma(\xi^2-\eta^2)$ and $n = \xi^2+\eta^2$, ξ is even and η is odd. It is evident that $\phi(q) \equiv 2 \pmod 4$ if $q \equiv 1 \pmod 4$. Thus

$$\tau(q) \equiv \phi(q) \equiv 2 \pmod 4. \qquad (2.7)$$

The lemma and the theorem follow easily as in the previous case since $\tau(2n) \equiv 0 \pmod 8$*.

5. PROOF OF THEOREM C.

LEMMA 5.† *If n is a non-residue of the prime p and $\sigma_a(n)$ is the sum of the ath powers of the divisors of n then*

$$\sigma_{(p-1)/2}(n) \equiv 0 \pmod p. \qquad (2.8)$$

PROOF. It is well known that if $(a, p) = 1$ and p is a prime number

$$a^{\frac{p-1}{2}} \equiv (a/p) \pmod p, \qquad (2.9)$$

where (a/p) is Legendre's quadratic residue symbol. Also conjugate divisors of a have like or unlike quadratic characters (mod p) according as a is a residue or a non residue mod p. In any case

$$\sigma_{(p-1)/2}(n) = \sum_{\delta | n} \delta^{\frac{p-1}{2}} \equiv \sum_{\delta | n} (\delta/p) \pmod p. \qquad (2.10)$$

When n is a non-residue the result easily follows.

Let us now write in the notation‡ of Ramanujan

$$P = 1-24\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\ldots\right) \qquad (2.11)$$

$$Q = 1+240\left(\frac{1^3x}{1-x}+\frac{2^3x^2}{1-x^2}+\ldots\right) \qquad (2.12)$$

* *Proc. Ind. Academy of Sciences*, 19 (1944), (146-8).

† More general results concerning the Congruence Properties of $\sigma_a(n)$ are given in a forthcoming paper in *Proc. Ind Academy of Sciences.*

‡ *Collected Papers of Srinivasa Ramanujan*, Cambridge, 1927.

$$R = 1-504\left(\frac{1^5x}{1-x}+\frac{2^5x^2}{1-x^2}+\ldots\right) \tag{2.13}$$

Ramanujan* has proved that

$$Q^3-R^2 = 1728\sum_1^\infty \tau(n)x^n \tag{2.14}$$

$$PQ-R = 720\sum_1^\infty n\sigma_3(n)x^n \tag{2.15}$$

$$Q^2-PR = 1008\sum_1^\infty n\sigma_5(n)x^n. \tag{2.16}$$

It is evident that $R\equiv 1 \pmod 7$ and $Q^2\equiv P \pmod 7$ so that it is easily seen that

$$\tau(n)\equiv n\sigma_3(n) \pmod 7. \tag{2.17}$$

Ramanujan's result that $\tau(n)\equiv 0 \pmod 7$ if $n\equiv 0, 3, 5, 6 \pmod 7$ follows at once by using Lemma 5.

The theorem itself follows by writing down the value of $\sigma_3(n)$ in terms of its prime factors.

* *Loc. Cit.*

A CONGRUENCE RELATION BETWEEN $\tau(n)$ AND $\sigma(n)$.

BY

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1. Proceeding on the lines of my paper(1) on the "Congruence Properties of $\tau(n)$", it can be shown that for any odd prime p,

$$\tau(p)\equiv p+1 \pmod 8$$
$$\equiv\sigma(p) \pmod 8,$$

where $\tau(n)$ is Ramanujan's function defined by the relation

$$\sum_{n=1}^\infty \tau(n)x^n = x\left\{(1-x)(1-x^2)\ldots\right\}^{24};$$

1. *Proc. Benares Math. Soc.* 1943.

and $\sigma(m)$ denotes the sum of the divisors of m.

Since[2] $\tau(p^k) = \tau(p^k)\, \tau(p^{k-1}) - p^{11}\, \tau(p^{k-2})$, $k > 2$, for all primes $p \geqslant 2$; when $(p, 2) = 1$, we have

$$\tau(p^2) \equiv (p+1)^2 - p^{11} \pmod 8,$$
$$\equiv p^2 + p + 1 \pmod 8, \text{ because } p^{11} \equiv p \pmod 8,$$
$$\equiv \sigma(p^2) \pmod 8.$$

Proceeding in this manner, we can show that

$$\tau(p^k) \equiv \sigma(p^k) \pmod 8,\ k > 1.$$

Since[2] $\tau(n)$ and $\sigma(n)$ are both multiplicative functions, *i. e.* $\tau(ij) = \tau(i)\tau(j)$ and $\sigma(ij) = \sigma(i)\sigma(j)$, when $(i, j) = 1$; we get $\tau(n) \equiv \sigma(n) \pmod 8$, provided $(n, 2) = 1$. (1)
When $(n, 2) = 2$, it is known that

$$\tau(n) \equiv 0 \pmod 8. \tag{2}$$

These results lead to the identical congruence

$$x\{(1-x)(1-x^2)\ldots\}^{24} \equiv \sum_{r=1}^{\infty} \frac{(2r-1)\, x^{2r-1}}{1-x^{4r-2}} \pmod 8.$$

2. Again if p be any prime other than 3, it is known[2] that

$$\tau(p) \equiv p+1 \pmod 3,$$
$$\equiv \sigma(p) \pmod 3.$$

Hence, proceeding as in §1, we get

$$\tau(n) \equiv \sigma(n) \pmod 3, \text{ when } (n, 3) = 1; \tag{3}$$

while $\tau(n) \equiv 0 \pmod 9$; when $(n, 3) = 3$. (4)

From (1) and (3), it follows that if $(j, n) = 1$,
then $\tau(n) \equiv \sigma(n) \pmod j$, $j = 2, 3, 4, 6, 8, 12$ or 24. (5)

In particular[3],

$$\tau(jm-1) \equiv 0 \pmod j,\ j = 3, 4, 6, 8, 12 \text{ or } 24;$$

and[1] $\tau(n) \equiv 1 \pmod 2$ if and only if n is an odd square.

2. Proved by Mordell, *Proc. Camb. Phil. Soc.* 19 (1919), 117-24. See also *Ramanujan* by Hardy, 1940, 161-3.

3. This follows from the Theorem: $\sigma(jm-1) \equiv 0 \pmod j$ for every $m \geqslant 1$ if and only if $j = 3, 4, 6, 8, 12$ or 24. See *Math. Student*, (1943), 33-5. Also proved by the author.

A THEOREM ON OPERATIONAL CALCULUS.

BY

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[Received 14 September 1945].

1. In this paper is given a theorem involving operational relations whose originals are self-reciprocal in the Hankel transform of order, ν say, that is they satisfy the integral equation

$$F(x)=\int_0^\infty \sqrt{xy}\,J_\nu(xy)\,F(y)\,dy,\ R(\nu)>-\tfrac{1}{2}. \qquad (1)$$

Following Hardy and Titchmarsh we shall call such functions R_ν. Since as $x\to\infty$, $J_\nu(x)\sim O[x^{-\frac{1}{2}}\cos\ (x-A)]$ and as $x\to 0$, $J_\nu(x)\sim O(x^\nu)$ (1) is convergent if $R(\nu)>-\frac{1}{2}$, and as $x\to\infty$, $F(x)\sim O(x^k)$, where $k<0$ and as $x\to 0$, $F(x)\sim o(x^\tau)$. where $R(\tau+\nu)>-\frac{3}{2}$.

2. THEOREM. *If* $f(p)\doteqdot h(x)$ *and* $x^\lambda h(x)$ *is* R_ν, *then*

$$f(\sqrt{p})\doteqdot\frac{\Gamma(\frac{1}{2}\nu-\frac{1}{2}\lambda+\frac{3}{4})}{2^{\lambda-\frac{1}{2}}\sqrt{\pi}\,\Gamma(\nu+1)}\,x^{-\frac{1}{2}\lambda-\frac{1}{4}}\int_0^\infty y^{\lambda-\frac{1}{2}}\,e^{-\frac{1}{2}xy^2}M_{\frac{1}{4}-\frac{1}{2}\lambda,\,\frac{1}{2}\nu}(xy^2)$$
$$\times h(y)\,dy, \qquad (2)$$

provided that as $y\to\infty$, $y^\lambda h(y)\sim o(y^\alpha)$, $\alpha<-1$ *and the functions involved are continuous in the unlimited range* $\leqslant 0$ *of the argument.*

PROOF. We know* that if $f(p)\doteqdot h(x)$ then

$$f(\sqrt{p})\doteqdot(\pi x)^{-\frac{1}{2}}\int_0^\infty \exp(-s^2/4x)\,h(s)\,ds.$$

Since $x^\lambda h(x)$ is R_ν, that is,

$$s^\lambda h(s)=\int_0^\infty y^\lambda h(y)\sqrt{sy}\,J_\nu(sy)\,dy \qquad (2.1)$$

we have

$$f(\sqrt{p})\doteqdot(\pi x)^{-\frac{1}{2}}\int_0^\infty e^{-s^2/4x}s^{\frac{1}{2}-\lambda}ds\int_0^\infty y^{\lambda+\frac{1}{2}}J_\nu(sy)h(y)\,dy. \quad (2.2)$$

* P. Humbert: *Le calcule Symbolique*; (Paris, 1934).

† Watson G. N.: *Theory of Bessel Functions*, (Camb., 1922), 393.

On changing the order of integration, which we shall presently justify, and using the formula†

$$\int_0^\infty t^{\mu-1}e^{-p^2t^2}J_\nu(at)dt = \frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}\mu)}{\Gamma(\nu+1)}\frac{p^{1-\mu}}{a}e^{-a^2/8p^2}$$
$$\times M_{\frac{1}{2}\mu-\frac{1}{2},\frac{1}{2}\nu}(a^2/4p^2),\ R(\mu+\nu)>0,$$

we arrive at the theorem. We now justify the inversion.

Considering that the interval in (2.1) converges (by hypothesis) and that as $|x|\to\infty$, $J_\nu(|x|)\sim O(|x|^{-\frac{1}{2}})$ it is easy to infer the absolute convergence of both the y-integral and the s-integral in (2.2), the former requiring that $\alpha<-1$. Also since as $x\to 0$, $M_{k,m}(x)\sim O(x^{m+\frac{1}{2}})$ and as $x\to\infty$, $M_{k,m}(x)\sim O(x^{-m-\frac{1}{2}}e^{\frac{1}{2}x})$ the integral in (2) is seen to be convergent under the same conditions as (2.1), viz. $R(\alpha-\nu)<\frac{1}{2}$, $R(\tau+\nu)>-\frac{3}{2}$, where $y^\lambda h(y)\sim o(y^\tau)$ as $y\to 0$.

Hence by de La Valee Poussin's theorem* changing the order of integration is legitimate if $\alpha<-1$.

3. Cor. If $\lambda=-\nu-\frac{1}{2}$, Kummer's function in (2) reduces to an exponential function and we have:

If $f(p)\doteqdot h(x)$ *and* $x^{-\frac{1}{2}-\nu}h(x)$ *is* R_ν, *then provided as* $y\to\infty$, $h(y)\sim O(y^\alpha)$ $\alpha<-1$,

$$f(\sqrt{p})\doteqdot\sqrt{\frac{2}{\pi}}\,(2x)^{\nu+\frac{1}{2}}\int_0^\infty h(y)\exp(-xy^2)dy,\ R(\nu)>-\tfrac{1}{2}.$$

This may be put in the alternative form that $f(\sqrt{p})$ and $x^{-\frac{1}{2}}h(\sqrt{x})$ have identical original and image save for a factor $\frac{1}{2}\sqrt{\pi}p^{\frac{1}{2}-\nu}$ in the latter, i.e.

If $f(p)\doteqdot h(x), f(\sqrt{p})\doteqdot\phi(x)$ *and* $x^{-\frac{1}{2}-\nu}h(x)$ *is* R_ν *then*

$$\tfrac{1}{2}\sqrt{\pi}p^{\frac{1}{2}-\nu}\phi(p)\doteqdot x^{-\frac{1}{2}}h(\sqrt{x}).$$

4. Illustration. We use a general form of an R_ν function. Consider the relation

$$h(x)\dagger\equiv x^\nu\int_0^\infty t^\nu e^{-\frac{1}{2}x^2t^2}\psi(t)dt\doteqdot p\int_0^\infty e^{-px}dx.\ x^\nu\int_0^\infty t^\nu e^{-\frac{1}{2}x^2t^2}\psi(t)dt,$$

* Bromwich, T. J. I'A: *Theory of Infinite Series*, (Macmillan, 1942), 504.

† For facility of verification the factor x^ν has been included in $h(x)$.

where $\psi(t) \equiv \psi(t^{-1})$ and as $t \to 0$, $\psi(t) \sim o(t^{\alpha})$ and as $t \to \infty$ $\psi(t) \sim o(t^{-\alpha})$, $R(\nu+\alpha) > -1$.

Since both the x-and t-integrals are absolutely convergent owing to the presence of the factor exp $(-\frac{1}{2}x^2t^2)$ in the integrand, the order of integration may be inverted as the resulting integral exists. Carrying out this inversion and using Whittaker's integral*

$$\int_0^\infty e^{-px} e^{-\frac{1}{2}x^2t^2} x^{2\mu-1} dx = \Gamma(2\mu) t^{-2\mu} \exp\left(\tfrac{1}{4}p^2/t^2\right) D_{-2\mu}(p/t),$$

$$R(\mu) > 0,$$

to evaluate the inner integral we find that, on putting u for $1/t$,

$$h(x) \doteqdot \Gamma(\nu+1)\, p \int_0^\infty u^{-1} \psi(u) D_{-\nu-1}(pu) e^{\frac{1}{4}p^2u^2} du \equiv f(p),$$

$$R(\nu+\alpha) > -1,\ R(\alpha) > 0.$$

Since† $x^{\frac{1}{2}} h(x)$ is R_ν, the theorem furnishes the relation

$$f(\sqrt{p}) \doteqdot \frac{1}{x} \int_0^\infty e^{-s} I_{\frac{1}{2}\nu}(s) (2s/x)^{\frac{1}{2}\nu} ds \int_0^\infty t^{\nu} \psi(t) \exp(-st^2/x) dt, \quad (4)$$

for $\qquad M_{0,\,m}(z) = \Gamma(m+1)\, 2^{2m} z^{\frac{1}{2}} I_\nu(\tfrac{1}{2}z).$

Inversion is again permissible as above in (4) owing to the presence of the factor exp $(-st^2/x)$ in the integrand. Evaluating the x-integral by means of the formula‡ (derivable from Formula (7) by putting $b = ci$),

$$\int_0^\infty e^{-at} I_\nu(ct) t^\nu dt = (2c)^\nu\, \Gamma(\nu+\tfrac{1}{2})/\sqrt{\pi}(a^2-c^2)^{\nu+\frac{1}{2}},$$

$$R(\nu) > -\tfrac{1}{2},\ a > c$$

there results the operational relation

$$p^{\frac{1}{2}} \int_0^\infty \psi(u) D_{-\nu-1}(\sqrt{p}u) e^{\frac{1}{4}pu^2} \frac{du}{u} \doteqdot \frac{x^{\frac{1}{2}\nu}}{\Gamma(\frac{1}{2}\nu+1)} \int_0^\infty \frac{\psi(t)dt}{t(t^2+2x)^{\frac{1}{2}\nu+\frac{1}{2}}},$$

where $\psi(u) \equiv \psi(1/u)$, $R(\nu+\alpha) > -1$, $R(\alpha) > 0$.

* Whittaker E. T. : *Proc. Lond. Math. Soc.* (1903), 417-27.

† Bailey, W. N. : *Jour. London Math. Soc.* 5 (1930), 92, 258.

‡ Watson, G. N. : *Loc. cit.* 386.

VERIFICATION. To verify the foregoing relation interpret either side of it by means of the relation*

$$\sqrt{p}\, e^{\frac{1}{4}p} D_{-2\mu}(\sqrt{p}) \doteqdot x^{\mu-\frac{1}{2}}/\Gamma(\mu+\tfrac{1}{2})(1+2x)^{\mu},\ R(\mu) > -\tfrac{1}{2}.$$

5. APPLICATIONS.

LEMMA:

$$f(p) \equiv i^n p\, D_n(ip)\, D_{-\nu-n-1}(p) \doteqdot n!\, x^{\nu} e^{-\frac{1}{2}x^2} T_\nu^n(x^2) \equiv h(x),$$
$$R(\nu) > -1.$$

PROOF. Replace μ by $r+\frac{1}{2}\nu+\frac{1}{2}$ in the relation† $\Gamma(2\mu) p\, e^{\frac{1}{4}p^2} D_{-2\mu}(p) \doteqdot x^{2\mu-1} e^{-\frac{1}{2}x^2}$; multiply both sides by $(-)^{r+n}/\Gamma(r+\nu+1)\,(n-r)!\,r!$ and sum from $r=0$ to $r=n$, the image function by the formula‡

$$i^{-n} D_n(ip)\, D_{-\nu-n-1}(p) = n!\, e^{\frac{1}{4}p^2}$$
$$\times \sum_{r=0}^{n} \frac{(-)^r \Gamma(\nu+1+2r)\, D_{-\nu-2r-1}(p)}{r!\,(n-r)!\,\Gamma(\nu+1+r)}$$

and the original by means of the formula

$$(-)^n T_\nu^n(u) = \sum_{r=0}^{n} (-u)^r / r!\,(n-r)!\,\Gamma(1+\nu+r).$$

RESULT 1. Since§ $x^{\frac{1}{2}} h(x)$ is R_ν, the theorem yields the relation

$$\sqrt{p}\, i^n D_n(i\sqrt{p})\, D_{-\nu-n-1}(\sqrt{p}) \doteqdot n!/x$$
$$\times \int_0^\infty \left(\frac{2t}{x}\right)^{\frac{1}{2}\nu} \exp\left(-t-\frac{t}{x}\right) I_{\frac{1}{2}\nu}(t)\, T_\nu^n\left(\frac{2t}{x}\right) dt, \quad (5.1)$$

Both the sides of (5.1) being analytic for $R(\nu) > -1$, (5.1) holds if $R(\nu) > -1$. The original of the left side is also obtained by starting with and following the procedure of the lemma. It is then found that

* Varma, R. S. : *Phil. Mag.* 22 (1936), 32.

† Varma, R. S. : *Proc. Camb. Phil. Soc.* 33 (1937), 210.

‡ Hari Shanker : *Proc. Benares Math. Soc.* (New Series), 2 (1940), 62.

§ Wilson, B. M. : *Messenger of Math.* 53 (1923-24), 157-60.

$$\sqrt{p}\, i^n D_n(i\sqrt{p})\, D_{-\nu-n-1}(\sqrt{p}) \doteq \frac{(-)^n n!}{2\sqrt{\lambda x}}$$

$$\times \sum_{r=0}^{n} \frac{(-)^r \Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}+r)}{r!(n-r)!\,\Gamma(\nu+1+r)} \left(\frac{4x}{1+2x}\right)^{r+\frac{1}{2}\nu+\frac{1}{2}}, R(\nu)>-1. \quad (5.2)$$

Equating the originals of (5.1) and (5.2) by Lerch's theorem*, we finally get

$$\int_0^\infty e^{-u(1+x)} I_{\frac{1}{2}\nu}(xu)\, T_\nu^h(2u)\, u^{\frac{1}{2}\nu}\, du = \frac{(-)^n(\tfrac{1}{2}x)^{\frac{1}{2}\nu}}{n!\,\Gamma(\tfrac{1}{2}\nu+1)\,(1+2x)^{\frac{1}{2}\nu+\frac{1}{2}}}$$

$$\times {}_2F_1\left(\begin{matrix}-n, \tfrac{1}{2}\nu+\tfrac{1}{2}\,; \\ \nu+1\,;\end{matrix}\; \frac{4x}{1+2x}\right), \; R(\nu)>-1.$$

6. RESULT 2. Lastly take the operational relation

$$f(p) \equiv p^{-2m} e^{\frac{1}{4}p^2} W_{m-\beta, m}(\tfrac{1}{2}p^2) \doteq \frac{2^{\beta-m} x^{2\beta}}{\Gamma(2\beta+1)}$$

$$\times {}_1F_1\left(\begin{matrix}\tfrac{1}{2}+\beta-2m\,; \\ \beta+1\,;\end{matrix}\; -\frac{x^2}{2}\right) \equiv h(x),\; R(\beta)>-\tfrac{1}{2},$$

which is easily deduced from a general relation obtained by Hari Shanker† and also from a more general one given by the author elsewhere, viz.

$$p^{\mu(k-\frac{1}{2})-\lambda} \exp(\tfrac{1}{2}p^\mu)\, W_{\frac{1}{2}-k, m}(p^\mu)$$

$$\doteq \sum_{r=0}^{\infty} \frac{\Gamma_2(r+k\pm m)\,(-)^r x^{\lambda+kr}}{\Gamma(1+\lambda+\mu r)\, r!\, \Gamma_2(k\pm m)}, R(\lambda)>-1,\; 1\leqslant \mu \leqslant 3,$$

where for brevity the product $\Gamma(k+m)\,\Gamma(k-m)$ is expressed by a new symbol $\Gamma_2(k\pm m)$.

Since‡ $x^{-\beta-2m}h(x)$ is $R_{\beta+\frac{1}{2}+2m}$ and the original of $f(\sqrt{p})$ can be obtained by termwise interpretation, an application of the theorem leads us to the integral

$$\int_0^\infty y^{-2m-\frac{3}{2}} M_{\nu,\nu}(xy^2)\, M_{\frac{1}{2}\beta-2m, \frac{1}{2}\beta}(\tfrac{1}{2}y^2)\, e^{-\frac{1}{2}(x+\frac{1}{2})y^2}\, dy$$

$$= \frac{\sqrt{\pi}\,\Gamma(2\nu+1)\, x^{\nu-2m}}{\Gamma(2\nu+\tfrac{1}{2})\,\Gamma(\beta+1)\, 2^{\frac{1}{2}\beta+3m+1}}\, {}_2F_1\left(\begin{matrix}\tfrac{1}{2}+\beta, \tfrac{1}{2}+\beta-2m\,; \\ \beta+1\,;\end{matrix}\; -2x\right),$$

where $\nu \equiv m+\tfrac{1}{2}\beta+\tfrac{1}{4}$ and $R(\beta+2m) > -1 < R(\beta)$.

* Lerch : *Acta Math.* 27 (1903), 339.

† Hari Shanker : *Jour. London Math. Soc.* 7 (1943), 63.

‡ Brij Mohan : *Quart. Jour. of Math.* 10 (1939), 256.

TWO THEOREMS ON SELF-RECIPROCAL FUNCTIONS.

BY

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1. The object of this note is to give two theorems which enable us to derive by processes of differentiation and integration new self-reciprocal functions from known ones. We shall call a function R_ν if it is self-reciprocal for J_ν transforms.

Hardy and Titchmarsh* have defined a class of functions $A(\omega, a)$, where $0 < \omega \leqslant \pi$, $a < \frac{1}{2}$, as functions which (i) are analytic functions of $x = re^{i\theta}$, regular in the angle A defined by $r > 0$, $|\theta| < \omega$ and (ii) are $o(|x|^{-a-\delta})$ for small x and $o(|x|^{a-1+\delta})$ for large x, for every positive δ and uniformly in any angle $|\theta| \leqslant \omega - \eta < \omega$. Concerning this class of functions they have given the theorem :—

A necessary and sufficient condition that a function $f(x)$ of $A(\omega, a)$ should be R_ν is that it should be of the form

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\ x^{-s}\psi(s)\ ds, \qquad (1.1)$$

where $\psi(s)$ is regular in $a < \sigma < 1-a$, $(s \equiv \sigma + it)$ (1.2)

satisfies the identical equation $\psi(s) \equiv \psi(1-s)$ (1.3)

and is $o\left\{ e^{(\frac{1}{4}\pi - \omega + \eta)\,|t|} \right\}$ (1.4)

for every positive η and uniformly in any strip interior to (1.2) and c is any value of σ in (1.2).

2. THEOREM 1. *Let E and E_1 stand for the operators*

$$x^{\lambda-1}\int_0^x dx\ x^{-\lambda} \text{ and } -x^{-\lambda}\int_0^x dx\ x^{\lambda-1}$$

respectively, $R(\lambda)$ lying between a and $1-a$. Then $E^m E_1^n + E_1^m E^n$ is an operator transforming an R_ν into another R_ν function, where $E^m E_1^n$ stands for the operation of E performed m times and that of E_1 performed n times in all in any order.

* G. H. Hardy and E. C. Titchmarsh : *Quart. Jour. of Math.* 1 (1930), 196-236.

PROOF. Let $f(x)$ belonging to the class $A(\omega, a)$ be R_ν. Then by the theorem of §1,

$$f(x) = \int_{c-i\infty}^{c+i\infty} P(s)\, x^{-s}\, ds,$$

where $$P(s) = \frac{1}{2\pi i} 2^{\frac{1}{2}s} \Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\, \psi(s),$$

$\psi(s)$ satisfying all the conditions given in §1.

Now $$Ef(x) = x^{\lambda-1} \int_0^x x^{-\lambda}\, dx \int_{c-i\infty}^{c+i\infty} P(s)\, x^{-s}\, ds \qquad (2.1)$$

$$= x^{\lambda-1} \int_{c-i\infty}^{c+i\infty} P(s)\, ds \int_0^x x^{-\lambda-s}\, dx = \int_{c-i\infty}^{c+i\infty} \frac{x^{-s}P(s)}{1-\lambda-s} ds.$$

The change of order in (2.1) is valid for $R(\lambda+s) < 1$ by de La Vallee Poussin's theorem* on account of the absolute convergence of both the $x-$ and $s-$ integrals, the latter since by virtue of the asymptotic behaviour†

$$\Gamma(\alpha+i\beta+i\gamma t) = O(e^{-\frac{1}{2}\pi|t\gamma|}\, t^{\alpha-\frac{1}{2}})$$

as $|t| \to \infty$

$$|P(s)\, x^{-s}| = o\,(e^{(-\omega+\eta)\,|t|}), \text{ for all } \eta > 0, \text{ by } (1.4).$$

Repeating the process it is easy to see that in general

$$E^m f(x) = \int_{c-i\infty}^{c+i\infty} \frac{P(s)\, x^{-s}}{(1-s-\lambda)^m} ds.$$

Similarly it may be shown that provided $R(s-\lambda) < 0$,

$$E_1^n f(x) = \int_{c-i\infty}^{c+i\infty} \frac{P(s)\, x^{-s}}{(s-\lambda)^n} ds.$$

It follows therefore that

$$E^m E_1^n f(x) = \int_{c-i\infty}^{c+i\infty} \frac{P(s)\, x^{-s}}{(s-\lambda)^n (1-s-\lambda)^m} ds,$$

provided $R(s-\lambda) < 0$ and $R(\lambda+s) < 1$, i.e. $a < R(\lambda) < 1-a$, the order of operations being indifferent. Hence

$$(E^m E_1^n + E_1^m E^n) f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s} \Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\, \psi_1(s)\, x^{-s}\, ds,$$

where (2.2)

* Bromwich : *Theory of Infinite Series*, (Macmillan, 1926), 503.

† E. T. Whittaker and Watson : *Modern Analysis*, 279.

$$\psi_1(s) \equiv \{ (1-s-\lambda)^{-m}(s-\lambda)^{-n} + (s-\lambda)^{-m}(1-s-\lambda)^{-n} \} \psi(s).$$

Since $\psi_1(s)$ satisfies all the conditions as $\psi(s)$ in §1, the theorem follows on comparing (2.2) with (1.1).

COR. When $m = n$ we have:

If $f(x)$ is R_ν then $E^m E_1^m f(x)$ is R_ν too, provided $a < R(\lambda) \nless 1-a$.

THEOREM II. *Let* $\Delta \equiv \lambda + xd/dx$ *and* $\Delta_1 \equiv \lambda - 1 - xd/dx$; *then* $\Delta^m \Delta_1^n + \Delta^n \Delta_1^m$ *is an operator transforming an R_ν into another R_ν function.*

The proof of this theorem follows the same lines as that of the previous theorem. It may be noted that

$$\Delta\Delta_1 \equiv (\lambda + xd/dx)\,(\lambda - 1 - xd/dx) \equiv \lambda(\lambda-1) - d/dx\,(x^2 d/dx).$$

Since the sum of two or more R_ν functions is itself R_ν, the operator $\Delta\Delta_1 - \lambda(\lambda-1)$ and consequently any polynomial in $d/dx\,(x^2 d/dx)$ will transform an R_ν into another R_ν function.

3. ILLUSTRATION. From the R_ν function $x^{\nu+\frac{1}{2}}e^{-\frac{1}{2}x^2}$ we obtain the R_ν function

$$\frac{d}{dx}\left[x^2 \frac{d}{dx}\left(x^{\nu+\frac{1}{2}} e^{-\frac{1}{2}x^2}\right)\right] = -(\nu+\tfrac{5}{4})\, x^{\nu+\frac{1}{2}} e^{-\frac{1}{2}x^2}$$

$$+ (\nu+1)(\nu+2)\left\{1 - \frac{2}{\nu+1}x^2 + \frac{x^4}{(\nu+1)(\nu+2)}\right\} x^{\nu+\frac{1}{2}} e^{-\frac{1}{2}x^2}.$$

Leaving out the first term which is R_ν we are led to the R_ν function

$$\frac{1}{2\Gamma(\nu+1)}\left\{1 - \frac{2x^2}{\nu+1} + \frac{x^4}{(\nu+1)(\nu+2)}\right\} x^{\nu+\frac{1}{2}} e^{-\frac{1}{2}x^2}$$

$$= T_\nu^2(x^2)\, x^{\nu+\frac{1}{2}} e^{-\frac{1}{2}x^2}.$$

Similarly we might obtain the general R_ν function $T_\nu^{2p}(x^2)\, x^{\nu+\frac{1}{2}} e^{-\frac{1}{2}x^2}$.

ON PERIODIC MEROMORPHIC FUNCTIONS.*

BY

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The first section of the note centres round the problem of constructing a periodic meromorphic function, with assigned poles and principal parts thereat, in a period strip. The second deals with rational functions of $\exp(2\pi iz/\lambda)$, the main theorem of the section giving a necessary and sufficient condition for a periodic meromorphic function to be such a rational function, in terms of its Nevanlinna characteristic function. Throughout it is supposed, for the sake of simplicity, that λ is a positive number.

I

1. Mittag-Leffler's theorem for the construction of a meromorphic function with given poles and principal parts at the poles, assumes, in the case of periodic functions, a special form (Theorem 1 *infra*) derived from

LEMMA 1. *Let* $\lambda > 0$, $z = x+iy$; $u = \exp(2\pi iz/\lambda)$, $\alpha = \exp(2\pi ia/\lambda)$. *Then*

$$P\left\{\cot \frac{\pi(z-a)}{\lambda}\right\} = \frac{\pi}{\lambda}\left[\sum_{\nu=0}^{p}(-1)^{\nu}\frac{A^{(\nu)}}{\nu!}\frac{d^{\nu}}{dz^{\nu}}\cot\frac{\pi(z-a)}{\lambda} - iA^{(0)}\right]$$

$$= \sum_{\nu=0}^{p}\frac{A^{(\nu)}(2\pi i/\lambda)^{\nu+1}}{(u-\alpha)^{\nu+1}}$$

$$= g\left(\frac{1}{u-\alpha}\right)$$

* This note has been suggested in the main by a paper of Dr. V. Ganapathy Iyer entitled "On Periodic Integral Functions" [1]. In fact it owes a great deal to the help and encouragement which I received from him at more than one stage in its preparation.

is a meromorphic function of period λ, *having poles at the set of points* $\{a+k\lambda\}$, $k=0, \pm 1, \ldots$, *and at* $z=a$, *the principal part*

$$G\left(\frac{1}{z-a}\right)=\sum_{\nu=0}^{p} \frac{A^{(\nu)}}{(z-a)^{\nu+1}}.$$

PROOF. The only assertion which may not be obvious is the identification of g with P. This follows from the fact that $g-P$, considered as a function of z, is regular for finite z and bounded on a sequence of circles of arbitrarily large radius with centre $z=0$. Hence $g-P$ is a constant which is readily seen to be zero by letting $y \rightarrow -\infty$.

THEOREM 1. *Let* $F(z)$ *be a meromorphic function with period* $\lambda(>0)$. *Let* $\{a_n^+\}$, $\{a_n^-\}$, $\{a_n^o\}$ *be the poles of* $F(z)$ *in the strip* $0 \leqslant x < \lambda$, *whose imaginary parts are positive, negative and zero respectively, the sequence* $\{a_n^o\}$ *being necessarily finite (say* N *in number). Let*

$$\alpha_n^+ = \exp(-2\pi i a_n^+/\lambda), \quad \alpha_n^- = \exp(2\pi i a_n^-/\lambda),$$
$$\alpha_n^o = \exp(2\pi i a_n^o/\lambda).$$

Let the principal parts at a_n^+, a_n^-, a_n^o *be respectively*

$$G_n^+\left(\frac{1}{a_n^+-z}\right), \quad G_n^-\left(\frac{1}{z-a_n^-}\right), \quad G_n^o\left(\frac{1}{z-a_n^o}\right),$$

in the notation of Lemma 1. *Then*

$$F(z) = H(z)+\sum_1^\infty \{P_n^+[\cot \pi(a_n^+-z)/\lambda]+Q_n^+(\exp-2\pi iz/\lambda)\}$$
$$+\sum_1^\infty \{P_n^-[\cot \pi(z-a_n^-)/\lambda]+Q_n^-(\exp 2\pi iz/\lambda)\}$$
$$+\sum_1^N P_n^o[\cot \pi(z-a_n^o)/\lambda],$$

where P_n^+, P_n^-, P_n^o *are related to* G_n^+, G_n^-, G_n^o *respectively as* P *is related to* G *in Lemma* 1; *each* $Q_n^\pm$ *is a polynomial in*

$\exp(\mp 2\pi i z/\lambda)$, *and can be chosen that* $\sum_{1}^{\infty}[P_n^{\pm}+Q_n^{\pm}]$ *is analytic for all finite* $z \neq a_n^{\pm}$; *and* H *is a periodic integral function.*

This follows immediately from Mittag-Leffler's theorem and Lemma 1.

The theorem which follows, with its corollary, is a simple case of Theorem 1, when the poles in a period strip are finite in number.

THEOREM 2. *Let* $F(z)$ *have the poles* a_n $(n=1, 2, \ldots N)$ *in the period strip* $0 \leqslant x < \lambda$, *the principal part at* a_n *being* $G_n[1/(z-a)]$, *in the notation of Lemma* 1. *Further, let* $M(\eta, F) = \max_{y=\eta} |F(z)|$ *be such that* $M(y, F) = O(1)$ *as* $|y| \to \infty$. *Then*

$$F(z) = C + \sum_{1}^{N} P_n\left(\cot \pi \frac{z-a_n}{\lambda}\right),$$

where C *is a constant.*

COR. *If, in Theorem* 2, *the poles are simple and* $F(z)$ *is an odd function, then*

$$F(z) = C + \frac{\pi}{\lambda}\left[A_0 \cot \frac{\pi z}{\lambda} + \sum_{n=1}^{p} A_n \cot\left(\pi \frac{z \pm a_n}{\lambda}\right) + A_{p+1} \cot \pi \frac{z-\lambda/2}{\lambda}\right],$$

where $a_0 = 0$, a_1, $a_2, \ldots, a_{p+1} = \lambda/2$ *are the poles in* $0 \leqslant x \leqslant \lambda/2$, $y \geqslant 0$.

This follows at once from Theorem 2 if we remember that, corresponding to every pole a_n in $0 \leqslant x < \lambda, y > 0$ and $0 < x < \lambda/2, y = 0$, we have a pole $-a_n + \lambda$ in $0 \leqslant x < \lambda$, $y < 0$ and $\lambda/2 < x < \lambda, y = 0$; the corresponding poles having the same residue A_n.

2. The corollary under Theorem 2 enables us to state conditions for the periodicity of

$$\zeta(z)=\frac{1}{z}+\sum_{1}^{\infty}\frac{2z}{z^2-\lambda_n^2},$$

where $\{\lambda_n\}$ is such that $0<\lambda_1<\lambda_2<\ldots,\ \lambda_n\to\infty\ (n\to\infty)$,

$$\sum_{1}^{\infty}\lambda_n^{-2} \text{ converges.}$$

Employing the definition of complementary A. P's [1, §5, 8f],* we can state

THEOREM 3. *A necessary and sufficient condition for $\zeta(z)$ to be a function with period $\lambda>0$ is that $\{\pm\lambda_n\}$ must be capable of being split up into (a) an even number of* [or (b) *an odd number of*] *distinct A. P's with common difference λ, (a) two of which are* [or (b) *one of which is*] *self-complementary while the remaining can be grouped into complementary pairs.*

PROOF. Let $\zeta(z)$ have the period λ and the poles $0, \lambda_1, \lambda_2, \ldots, \lambda_p, \lambda_{p+1}=\lambda/2$ in $0\leqslant x\leqslant\lambda/2$, every pole being simple with residue unity. Since $\zeta(z)$ is an odd function which is bounded in any portion of the plane from which neighbourhoods of the poles are excluded, we have, by the corollary under Theorem 2,

$$\zeta(z)=C+\frac{\pi}{\lambda}\cot\pi\frac{z}{\lambda}+\frac{\pi}{\lambda}\sum_{n=1}^{p}\cot\left(\pi\frac{z\pm\lambda_n}{\lambda}\right)+\frac{\pi}{\lambda}\cot\left(\pi\frac{z-\lambda/2}{\lambda}\right),$$

whence letting $z\to 0$ we find that $C=0$. The poles of the expression on the right side give A. P's as in the condition (*a*) of our theorem. Thus the condition (*a*) is necessary when $\lambda_n=\lambda/2$ for a particular n and obviously (*b*) is necessary when $\lambda_n\neq\lambda/2$ for any n.

To prove that either condition is sufficient, choose from one of every pair of complementary A. P's one

* Two infinite A. P's $\{a+nd\}$ and $\{b+nd\}$, $n=0, \pm 1, \pm 2, \ldots$, with the same common difference, are complementary when one consists of the negatives of the terms of the other. If two complementary A. P's are identical, each is called a self-complementary A. P.

number in $0 < x < \lambda/2$, thus obtaining the set $\lambda_1, \lambda_2, \ldots, \lambda_p$. To this set add $0, \lambda/2$ from the two self-complementary A.P's in case (*a*), or 0 from the single self-complementary A.P. in case (*b*). Then

$$Z(z) = \frac{\pi}{\lambda}\cot \pi\frac{z}{\lambda} + \frac{\pi}{\lambda}\sum_{n=1}^{p}\cot\left(\pi\frac{z\pm\lambda_n}{\lambda}\right) + \frac{\pi}{\lambda}\cot\left(\pi\frac{z-\lambda/2}{\lambda}\right)$$

—with the last term present or absent according as it is case (*a*) or case (*b*) that arises—has in the entire plane the same poles, and the same principal parts at the poles, as $\zeta(z)$. Hence $Z(z)-\zeta(z)$ is regular in the whole plane and therefore a constant which is seen to be zero by making $z \to 0$. And so $\zeta(z) = Z(z)$ has period λ.

COR. *A necessary and sufficient condition for $\zeta(z)-1/z$ to be a function with period $\lambda > 0$ is, that $\{\pm\lambda_n\}$ must be capable of being split up into (a) an odd number [or (b) an even number] of distinct A.P's with common difference λ, (a) one of which is self-complementary while the remaining [or (b) which] can be grouped into complementary pairs.*

REMARK. It will be noticed that Theorem 3 is not a deduction from the conditions for the periodicity of $\sigma(z) = z\prod_{1}^{\infty}(1-z^2/\lambda_n^2)$ [1, Theorem 2-F]. For, although the periodicity of $\sigma(z)$ implies that of $\sigma'(z)/\sigma(z) = \zeta(z)$, the converse is not necessarily true.

3. The following theorem shows how a relaxation of the condition imposed on M in Theorem 2 leads, in certain cases, to a function $F(z)$ of non-finite order.

THEOREM 4. *Let $F(z)$ be a meromorphic function with period λ and having a finite number of poles and zeros in $0 \leqslant x < \lambda$. Let*

$$\overline{\lim_{|y|\to\infty}} \frac{\log\log M(y,F)}{|y|} = \omega < \infty.$$

Then

$$F(z) = e^{H(z)} R\,(\exp 2\pi i z/\lambda),$$

where $R(u)$ *is a rational function of* u, $H(z)$ *is a trigonometric polynomial in* z *of degree not exceeding* $\omega\lambda/2\pi$ *(and therefore reducing to a constant when* $\omega < 2\pi/\lambda$*).*

The proof of the theorem is along the same lines as that of its particular case, $\omega < 2\pi/\lambda$ [2, IV Abschn, Nr. 168].

II

4. The following lemma differs from Theorem 4 in that it expressly assumes $F(z)$ to be a function of finite order.

Lemma 2. *Let* $F(z)$ *be a meromorphic function with period* λ, *of finite order, and having a finite number of poles and zeros in* $0 \leqslant x < \lambda$. *Then*

$$F(z) = R\,(\exp 2\pi iz/\lambda)$$

and its characteristic function $T(r, F)$ *satisfies the condition*

$$\overline{\lim_{r\to\infty}}\ \frac{T(r, F)}{r} < \infty .$$

Proof. Clearly

$$F(z) = e^{H(z)}\, R\,(\exp 2\pi iz/\lambda),$$

where $H(z)$ is an integral periodic function. Also it follows from the extension of Hadamard's factorization theorem to meromorphic functions that $H(z)$ reduces to a constant. To complete the proof we observe that $F(z)$ satisfies, in Nevanlinna's notation [3, §§8.9, 8.91, 8.92] the conditions

$$m(r, \infty) = O(r),\ n(r, \infty) = O(r),\ N(r, \infty) = O(r)$$

and so

$$T(r) = m(r, \infty) + N(r, \infty) = O(r).$$

Theorem 5. *If* $F(z)$ *is a meromorphic function of finite order and period* λ, *then a necessary and sufficient condition for* $F(z)$ *to have a finite number of poles and zeros in* $0 \leqslant x < \lambda$, *is*

$$\underline{\lim}_{r\to\infty}\ \frac{T(r, F)}{r} < \infty .$$

Further

$$F(z) = R(\exp 2\pi iz/\lambda).$$

Proof. If $F(z)$ is of finite order and has a finite number of poles and zeros in a period strip, Lemma 2 shows that $\overline{\lim}\ T(r)/r < \infty$ holds, and *a fortiori*, $\underline{\lim}\ T(r)/r < \infty$.

If $F(z)$ satisfies the condition $\underline{\lim}\ T(r)/r < \infty$, it follows from the relation

$$N(r,\ a) \leqslant T(r) + O(1)$$

that

$$\underline{\lim}\ N(r,\ a)/r < \infty.$$

Now applying to the function $N(r, a)$:

$$N(r,\ a) = \int_0^r \frac{n(x,\ a) - n(0,\ a)}{x}\, dx + n(0,\ a) \log r,$$

a familiar argument, [*e.g.* 3, §8.21,] we obtain

$$\underline{\lim}\ n(r,\ a)/r < \infty.$$

After this, it is easily established, arguing as in the proof of Theorem 2B in [1], that the number of zeros of $F(z) - a$ in a period strip is finite. The given condition thus ensures that $F(z)$ shall have a finite number of poles and zeros in a period strip.

Cor. *If a periodic meromorphic function $F(z)$, of finite order, has more than a finite number of poles and zeros in a period strip, then* $\lim\ T(r,\ F)/r = \infty$.

5. The remaining theorems concern the positions of the zeros and the poles, in a period strip, of a rational function of $\exp\,(2\pi iz/\lambda)$. The second half of Theorem 6 is an immediate consequence of Jensen's formula, while Theorems 7-A, 7-B are transformations of the following result [2, III Abschn, Nr. 296].

Lemma 3. *Let $f(u)$ have zeros $u_1, \ldots u_m$ and poles $\alpha_1, \ldots \alpha_p$ inside the circle $|u| = R$ and be analytic elsewhere inside and on the circle. Let $|f(u)| = c > 0$ on $|u| = R$. Then*

$$f(u) = ce^{i\omega} \prod_{n=1}^{m} \frac{R(u-u_n)}{R^2 - \bar{u}_n u_n} \prod_{n=1}^{p} \frac{R^2 - \bar{\alpha}_n u}{R(u-\alpha_n)},$$

where ω *is real, and a factor corresponding to a zero or a pole appears a number of times equal to its order.*

THEOREM 6. *A rational function* $F(z)$ *of* $\exp(2\pi i z/\lambda)$ *has in the entire period strip* $0 \leqslant x < \lambda$ *as many poles as it has zeros, every pole or zero being counted a number of times equal to its order. If further* $F(z) \to l^+$ *as* $y \to \infty$ *and* $F(z) \to l^-$ *as* $y \to -\infty$, *then*

$$\operatorname{Im}(z_1+z_2+\ldots+z_m-a_1-a_2-\ldots-a_m) = (\lambda/2\pi)\log|l^-/l^+|,$$

where $z_1, z_2, \ldots z_m$ *are the zeros and* $a_1, a_2, \ldots a_m$ *the poles.*

THEOREM 7-A. *If a meromorphic function with period* λ *has a constant modulus on* $y = y_0$ *and if its zeros and poles in the region:* $0 \leqslant x < \lambda, y > y_0$, *are given finite sets, then it is a rational function of* $\exp(2\pi i z/\lambda)$ *determined save for a factor whose modulus is unity. Further its zeros and poles in* $y < y_0$ *are images with respect to* $y = y_0$ *of its poles and zeros in* $y > y_0$.

This is the result of applying to Lemma 3 the transformation $u = \exp(2\pi i z/\lambda)$.

THEOREM 7-B. *If a meromorphic function with period* λ *has the same constant modulus on* $x = 0$ *and* $x = \lambda/2$ *and if its zeros and poles in* $0 < x < \lambda/2$ *are given finite sets, then it is a rational function of* $\exp(2\pi i z/\lambda)$ *determined save for a factor whose modulus is unity. Further its zeros and poles in* $\lambda/2 < x < \lambda$ *can be paired off with its poles and zeros in* $0 < x < \lambda/2$ *so that a zero and a pole of every pair are symmetric with respect to* $z = \lambda/2$.

This is the result of transforming Lemma 3 by setting

$$u = (1+ie^{2\pi i z/\lambda})/(1-ie^{2\pi i z/\lambda}),\ R = 1.$$

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MADRAS

ON THE ABERRANCY CURVE.

BY

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1. The locus of the centres of conics having contact of the third order with a given plane curve at a given point is a straight line through the point, called the line of aberrancy. The centre of the osculating conic, i.e. the conic having contact of the fourth order lies on this line and is called the centre of aberrancy. Its locus, as the given point moves on the curve, is the curve of aberrancy. A few properties associated with these ideas are well known, the most interesting of which is the property* that the radius of curvature of the aberrancy curve contains as a factor the 'Mongian', the vanishing of which gives the differential equation of all conics. This property is due to the late Sir Ashutosh Mukherjee (Mukhopadhayay). In this note, some further properties of the aberrancy curve are discussed.

2. Taking the origin as a point on the curve $x=\theta$, $y=f(\theta)$, the new axes being parallel to the given ones, let us express the equation of the curve in the form

$$y=px+qx^2/2!+rx^3/3!+sx^4/4!+\ldots, \qquad (1)$$

where p, q, r, s are the values of the successive differential coefficients of y w.r.t. x at the origin. The equation of the osculating conic at the origin must be of the form $y-px=lx^2+mxy+ny^2$. Substituting for y from (1) and writing down the conditions for contact of the fourth order, we obtain on elimination of l, m, n, the equation of the osculating conic in the form

$$18q^3(y-px)=[p^2(3qs-4r^2)+3q^2(3q^2-2pr)]\,x^2 \\ +2[3q^2r-p(3qs-4r^2)]\,xy+(3qs-4r^2)y^2.$$

This can be written in the following elegant form

* Edwards: *Differential Calculus*, pp. 280-3 and p. 436.

$$[(3q^2-pr)\,x+ry]^2=(y-px)\,[(3qs-5r^2)\,(px-y)+18q^3].\quad(2)$$

Equation (2) shows that the lines $y=px$ and $(3qs-5r^2)(px-y)+18q^3=0$ are parallel tangents to the osculating conic and that $(3q^2-pr)x+ry=0$ is the chord joining their points of contact. Hence this chord passes through the centre of the conic, and is therefore the line of aberrancy. We have thus the equation of the line of aberrancy at the origin

$$(3q^2-pr)x+ry=0.\qquad(3)$$

We easily obtain the co-ordinates of the centre of aberrancy as

$$x=-3qr/(3qs-5r^2),\, y=3q\,(3q^2-pr)/(3qs-5r^2).\qquad(4)$$

If in (3) and (4) we write $x-\theta$ and $y-f(\theta)$ instead of x and y, and consider p, q, r, s as successive derivatives of $f(\theta)$, viz. $p=f'(\theta)$, $q=f''(\theta)$, etc. we obtain the line of aberrancy and the centre of aberrancy at any point θ on the given curve. Let us now consider the envelope of the line

$$(3q^2-pr)\cdot(x-\theta)+r\,[y-f(\theta)]=0.\qquad(5).$$

Differentiating w.r.t. θ and solving, we obtain for $x-\theta$ and $y-f(\theta)$ the values on the right side of (4). Hence we have

THEOREM I. *The curve of aberrancy is also the envelope of the line of aberrancy.*

If $q=0$, the point is an inflexional point on the curve. It follows from (4), that the aberrancy curve passes through the points of inflexion of the given curve. Further, by differentiating $x=\theta-3qr/(3qs-5r^2), y=f(\theta)-3q(3q^2-pr)/(3qs-5r^2)$, we find $dx/d\theta=r.T/(3qs-5r^2)^2$; $dy/d\theta=(pr-3q^2).T/(3qs-5r^2)^2$ and these give the dy/dx of the aberrancy curve to be $(pr-3q^2)/r$, T being the Mongian $40r^3-45qrs+9q^2t$. When $q=0$, the dy/dx of the aberrancy curve is also p; we have also d^2y/dx^2 for the aberrancy curve $=q(3qs-5r^2)^3/r^3.T$ and this vanishes when $q=0$. From these expressions we arrive at the following

THEOREM 2. *The points of inflexion of a given curve are also points of inflexion on its aberrancy curve, and the inflexional tangents at them are the same for both.*

Again, the d^2y/dx^2 of the aberrancy curve also vanishes when $3qs-5r^2=0$, and this leads to the following

THEOREM 3. *If at a point on a given curve, the osculating conic becomes a parabola, the corresponding point on the aberrancy curve is a point of inflexion at infinity and the line at infinity is the inflexional tangent.*

3. The radius of curvature of the aberrancy curve is readily calculated to be $\rho = T.[r^2+(3q^2-pr)^2]^{\frac{3}{2}}/q(3qs-5r^2)^3$, where T is the Mongian.

The vanishing of ρ with T leads to the theorem mentioned in §1. Now $T=0$ is found to be the condition that the osculating conic should have contact of the fifth order. At such a point, therefore, $\rho=0$. Using this and the fact that the tangent to the evolute is normal to the involute, we may enunciate the following

THEOREM 4. *The evolute of the aberrancy curve of a given curve cuts the aberrancy curve orthogonally at the centre of a conic having contact of the fifth order with the given curve.*

A GENERALIZATION OF WILSON'S THEOREM.

BY

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The object of this paper is to obtain a generalization of Wilson's theorem which includes as a particular case, Gauss's generalization. We prove the following

THEOREM 1. *If n is an odd number and α runs through the nth power residues of m* which are prime to m, then*

* Here, as well as in the rest of the paper, when we speak of residues mod m we shall not consider numbers belonging to the same residue class as different.

$$\Pi\alpha \equiv \pm 1 \bmod m,$$

where the lower sign holds for $m = 4$, p^a, or $2p^a$, p being an odd prime, and the upper sign for all other values of m.

Gauss's generalization is obviously the case for $n = 1$. The corresponding result for even values of n is less simple. We may prove

THEOREM 2. *If n is even and equal to $2^\nu n_1$, where n_1 is odd, and α runs through the nth power residues of m which are prime to m, then*

$$\Pi\alpha \equiv -1 \bmod m$$

if m is of the form p^a or $2p^a$, where p is a prime of the form $2^{\nu+1}\lambda+1$;

$$\Pi\alpha \equiv 1 \bmod m$$

if either m has at least two different prime factors of the form $2^{\nu+1}\lambda+1$, or m has at least one such prime factor and a factor 2^μ, where $\mu > \nu+2$, or m is odd and has no prime factor of the form $2^{\nu+1}\lambda+1$;

$$\Pi\alpha \equiv \begin{cases} 2^{\mu-1}+1 \bmod 2^\mu, \\ 1 \bmod m_1, \end{cases}$$

if m is equal to $2^\mu m_1$, where $\mu \leqslant \nu+2$, and m_1 is odd and has no prime factor of the form $2^{\nu+1}\lambda+1$; and, finally

$$\Pi\alpha \equiv \begin{cases} -1 \bmod p^a \\ +1 \bmod m_2 \end{cases}$$

if m is equal to $p^a m_2$, where p is a prime of the form $2^{\nu+1}\lambda+1$, and m_2 has no such prime factors and is odd.

We require the following lemmas.

LEMMA 1. *If a is prime to m and is an nth power residue of m, then the inverse of a, i.e. the number $\bar{a}$ which satisfies the relation*

$$a\bar{a} \equiv 1 \bmod m$$

is also an n th power residue of m.

For, let $a \equiv x^n \bmod m$.

Then obviously x is also prime to m and so there exists a number $\bar{x}$ such that

$$x\bar{x} \equiv 1 \bmod m.$$

Hence

$$a\bar{x}^n \equiv x^n \bar{x}^n \equiv 1 \bmod m$$

so that

$$\bar{x}^n \equiv \bar{a} \bmod m.$$

LEMMA 2. *If p is an odd prime and*

$$x^2 \equiv 1 \bmod p^a$$

then either

$$x \equiv 1 \bmod p^a,$$

or

$$x \equiv -1 \bmod p^a.$$

For,

$$x^2 \equiv 1 \bmod p^a$$

implies

$$(x-1)(x+1) \equiv 0 \bmod p^a;$$

and both $x-1$ and $x+1$ cannot be multiples of p since their difference is not a multiple of p.

LEMMA 3. *The solutions of*

$$x^2 \equiv 1 \bmod 2^\mu$$

are 1, $2^{\mu-1}-1$, $2^{\mu-1}+1$, $2^\mu-1$ *if* $\mu > 2$, 1 *and* 3 *if* $\mu = 2$, *and* 1 *if* $\mu = 1$.

The last two cases are obvious. We may therefore assume μ to be greater than 2. We have

$$(x-1)(x+1) \equiv 0 \bmod 2^\mu.$$

It follows that both $x-1$ and $x+1$ are even. Let us write

$$x-1 = 2^r y, \quad x+1 = 2^s z.$$

Then

$$2 = (x+1)-(x-1) = 2^s z - 2^r y,$$

and so the least of r, s is 1 and the other is therefore at least $\mu-1$. Taking $s = 1$ we have

$$x-1 \equiv 0 \bmod 2^{\mu-1}.$$

The only values of x subject to this condition which satisfy

$$x^2 \equiv 1 \bmod 2^\mu$$

are obviously 1 and $2^{\mu-1}+1$. Similarly, taking $r = 1$ we get the solutions $2^{\mu-1}-1$ and $2^\mu-1$.

LEMMA 4. *If n is odd, the equation* $x^n \equiv \pm 1 \bmod m$ *has a solution.*

In fact we have only to take $x \equiv \pm 1 \bmod m$.

LEMMA 5. *If n is odd the equation*
$x^n \equiv 2^{\mu-1} \pm 1 \bmod 2^\mu$ *can be satisfied by* $x = 2^{\mu-1} \pm 1$.

For, let $n = 2q+1$. Then

$$(2^{\mu-1} \pm 1)^{2q} = \{ (2^{\mu-1} \pm 1)^2 \}^q \equiv 1 \bmod 2^\mu,$$

and so

$$(2^{\mu-1} \pm 1)^{2q+1} \equiv 2^{\mu-1} \pm 1 \bmod 2^\mu.$$

LEMMA 6. *If n is even and equal to $2^\nu n_1$, where n_1 is odd then* $x^n + 1 \equiv 0 \bmod m$ *has a solution if and only if* $x^{2^\nu} + 1 \equiv 0 \bmod m$ *has a solution.*

For,

$$x^n + 1 \equiv 0 \bmod m$$

implies

$$(x^{n_1})^{2^\nu} + 1 \equiv 0 \bmod m.$$

Conversely

$$x^{2^\nu} + 1 \equiv 0 \bmod m$$

gives

$$x^n \equiv (x^{2^\nu})^{n_1} \equiv (-1)^{n_1} \equiv -1 \bmod m.$$

LEMMA 7. *If p is an odd prime, then*

$$x^{2^\nu} + 1 \equiv 0 \bmod p^a$$

has a solution if and only if p is of the form $2^{\nu+1}\lambda + 1$.

First of all we note that

$$x^{2^\nu} + 1 \equiv 0 \bmod p^a$$

implies

$$x^{2^\nu} + 1 \equiv 0 \bmod p;$$

and that if

$$x^{2^\nu} + 1 \equiv 0 \bmod p,$$

so that

$$x^{2^\nu} = -1 + kp,$$

then

$$(x^{p^{a-1}})^{2^\nu} = (-1+kp)^{p^{a-1}}$$
$$\equiv -1 \bmod p^a.$$

This means that

$$x^{2^\nu}+1 \equiv 0 \bmod p^a$$

has a solution if and only if

$$x^{2^\nu}+1 \equiv 0 \bmod p$$

has a solution. Next, if p is of the form $2^{\nu+1}\lambda+1$, then

$$\varphi(p) = p-1 = 2^{\nu+1}\lambda$$

so that, by a well-known theorem

$$x^{2^{\nu+1}}-1 \equiv 0 \bmod p$$

has exactly $2^{\nu+1}$ solutions of which 2^ν are solutions of

$$x^{2^\nu}+1 \equiv 0 \bmod p.$$

Finally, if

$$x^{2^\nu}+1 \equiv 0 \bmod p,$$

then

$$x^{2^{\nu+1}} \equiv 1 \bmod p.$$

If p is of the form $2^\mu\lambda+1$, where λ is odd, we have $\varphi(p) = p-1 = 2^\mu\lambda$; and, therefore,

$$x^{2^{\nu+1}} \equiv 1 \bmod p,\ x^{2^\nu} \not\equiv 1 \bmod p$$

imply that $2^{\nu+1}$ is a factor of $2^\mu\lambda$ or that $\nu+1 \leqslant \mu$.

LEMMA 9. *If* $\nu > 0$, *then*

$$x^{2^\nu}+1 \equiv 2^{\mu-1} \bmod 2^\mu$$

has no solutions if $\mu > 2$.

LEMMA 10. *If* $\nu > 0$, *then*

$$x^{2^\nu}-1 \equiv 2^{\mu-1} \bmod 2^\mu$$

has a solution if and only if $\mu > \nu+2$.

First, let us suppose that $\mu > \nu+2$.

Then, the formula

$$x^{2^\nu}-1 = (x-1)(x+1)(x^2+1)\dots(x^{2^{\nu-1}}+1)$$

shows that if we write $x = 2^{\mu-\nu-1}+1$, then $x^{2^\nu}-1$ is of the form $2^{\mu-1}\lambda$ where λ is odd, so that

$$x^{2^\nu}-1\equiv 2^{\mu-1} \bmod 2^\mu$$

is satisfied by $x=2^{\mu-\nu-1}+1$.

Next, let us suppose that

$$x^{2^\nu}-1\equiv 2^{\mu-1} \bmod 2^\mu$$

is satisfied for some value of x. x being necessarily odd, let it be equal to $2^k\lambda+1$ where λ is odd. Substituting this value in the relation

$$x^{2^\nu}-1=(x-1)(x+1)(x^2+1)(x^{2^2}+1)\ldots\ldots(x^{2^{\nu-1}}+1)$$

we see that $2^{k+\nu}$ is a factor of $x^{2^\nu}-1$ if $k>1$ and that $2^{\nu+2}$ is a factor if $k=1$. It follows that $\mu-1$ is at least equal to $\nu+2$.

LEMMA II. *If we denote the number of nth power residues of m whose squares are* 1 mod *m, by* $\nu(m)$ *and their product by* $\mu(m)$ mod m, *then, for m prime to m′,*

$$\nu(m)\ \nu(m')=\nu(m\,m')$$

and

$$\mu(mm')\equiv\{\mu(m)\}^{\nu(m')} \bmod m,$$

$$\mu(m\,m')\equiv\{\mu(m')\}^{\nu(m)} \bmod m'.$$

Let α_i, β_j represent residues of m and m' of the type mentioned above so that for some x, y,

$$x^n\equiv\alpha_i \bmod m,\ \alpha_i^2\equiv 1 \bmod m,$$

and $$y^n\equiv\beta_j \bmod m',\ \beta_j^2\equiv 1 \bmod m'.$$

m being prime to m', there exists by the Chinese remainder theorem a unique number z mod $m\,m'$ such that

$$z\equiv x \bmod m,\ z\equiv y \bmod m'.$$

It follows that if

$$z^n\equiv\gamma_{ij} \bmod m\,m',$$

then

$$\gamma_{ij}\equiv\alpha_i \bmod m,\ \gamma_{ij}\equiv\beta_j \bmod m',$$

so that

$$\gamma_{ij}^2\equiv\alpha_i^2\equiv 1 \bmod m,\ \gamma_{ij}^2\equiv\beta_j^2\equiv 1 \bmod m',$$

and so

$$\gamma_{ij}^2 \equiv 1 \bmod mm'.$$

Therefore γ_{ij} is a residue of mm' of the given type. Given α_i, β_j it is obvious that α_{ij} mod mm' is uniquely determined.

Conversely with each residue γ of mm' of the given type may be associated a unique residue of m and a unique residue of m', also of the given type. For, if

$$z^n \equiv \gamma \bmod mm', \; \gamma^2 \equiv 1 \bmod mm',$$

and $\gamma \equiv \alpha \bmod m, \; \gamma \equiv \beta \bmod m'$,

then $z^n \equiv \alpha \bmod m, \; \alpha^2 \equiv 1 \bmod m,$

and $z^n \equiv \beta \bmod m', \; \beta^2 \equiv 1 \bmod m'.$

Thus we see that the number of γ_{ij}'s is equal to the product of the number of α_i's and the number of β_j's, i.e.

$$\nu(mm') = \nu(m)\,\nu(m').$$

Again,

$$\mu(mm') \equiv \prod_{j=1}^{\mu(m')} \prod_{i=1}^{\nu(m)} \gamma_{i,j} \bmod mm',$$

and so, $\mu(mm') \equiv \left(\prod_i \alpha_i\right)^{\nu(m')} \equiv \left\{ \mu(m) \right\}^{\nu(m')}$ mod m and, similarly,

$$\mu(mm') \equiv \left\{ \mu(m') \right\}^{\nu(m)} \bmod m'.$$

We shall now proceed to prove Theorems 1 and 2.

By Lemma 1 we see that if α is in the set of nth power residues of m, then $\bar{\alpha}$ is also in the set. Hence we can pair off the residues α, with certain exceptions, in such a way that the product of the numbers of any pair is 1 mod m. The exceptions are those residues α for which $\bar{\alpha} \equiv \alpha$. It follows that in proving the theorems we may restrict the choice of the residues to those α's which satisfy the additional condition $\alpha^2 \equiv 1 \bmod m$.

PROOF OF THEOREM 1. Let n be an odd number, $m = 2^{\mu} p_1^{a_1} \ldots p_r^{a_r}$, where $p_1, p_2, \ldots, p_r$ are distinct odd primes. Then

$$\alpha^2 \equiv 1 \bmod m$$

implies

$$\alpha^2 \equiv 1 \mod 2^{\mu},$$
$$\alpha^2 \equiv 1 \mod p_i^{a_i} \ (i = 1, 2, \ldots r).$$

n being odd, it follows from Lemmas 2, 3, 4 and 5 that α is an n th power residue of each of 2^{μ}, $p_i^{a_i}$ $(i = 1, 2, \ldots r)$. Hence there exist x_i $(i = 0, 1, 2, \ldots r)$ such that

$$\alpha \equiv x_0^n \mod 2^{\mu}, \ \alpha \equiv x_i^n \mod p_i^{a_i} \ (i = 1, 2, \ldots r).$$

Now, by the Chinese remainder theorem there exists a unique z mod m such that

$$z \equiv x_0 \mod 2^{\mu}, \ z = x_i \mod p_i^{a_i} \ (i = 1, 2, \ldots r).$$

Therefore

$$z^n \equiv \alpha \mod (2^{\mu}, p_1^{a_1}, \ldots, p_r^{a_r}),$$

and so

$$z^n \equiv \alpha \mod m.$$

It follows that, n being odd, every solution of

$$\alpha^2 \equiv 1 \mod m$$

is also an n th power residue of m.

When m is either 4 or of the form p^a where p is an odd prime, Lemmas 2 and 3 show that the product of the solutions of

$$\alpha^2 \equiv 1 \mod m$$

is -1 mod m.

When m is of the form $2p^a$, we have, by Lemmas 11 and 2,

$$\pi\alpha \equiv 1^2 \equiv -1 \mod 2, \ \pi\alpha \equiv -1 \mod p^a$$

and hence

$$\pi\alpha \equiv -1 \mod 2p^a,$$

where α runs through the solutions (two in number) of

$$\alpha^2 \equiv 1 \mod 2p^a.$$

By Lemma 2, since

$$\alpha^2 \equiv 1 \mod p^a$$

has exactly two solutions and the product of these solutions is -1 mod p^a, we see, by Lemma 11, that

$$\alpha^2 \equiv 1 \bmod p_1^{a_1} p_2^{a_2}$$

has four solutions p_1, p_2 being distinct odd primes, and that the product of these solutions is $(-1)^2 = 1 \bmod p_1^{a_1}$ as well as mod $p_2^{a_2}$ and hence also mod. $p_1^{a_1} p_2^{a_2}$. From this we see by another application of Lemma 11, that

$$\alpha^2 \equiv 1 \bmod p_1^{a_1} p_2^{a_2} p_3^{a_3}$$

has 2^3 solutions and that the product of these solutions is 1 mod $p_1^{a_1} p_2^{a_2} p_3^{a_3}$. Proceeding in this manner we see that

$$\alpha^2 \equiv 1 \bmod p_1^{a_1} \dots p_r^{a_r} \, (r \geqslant 2),$$

where $p_1, p_2, \dots$ are distinct primes, has exactly 2^r solutions and that the product of these solutions is 1 mod $p_1^{a_1} \dots p_r^{a_r}$.

Finally, when m is of the form

$$2^\mu p_1^{a_1} \dots p_r^{a_r} \begin{pmatrix} r \gtreqless 1, \ \mu \gtreqless 2 \\ \text{or } r = 0, \ \mu > 2 \end{pmatrix}$$

we see, in the same manner, by Lemmas 2, 3 and 11, that the product of the solutions of

$$\alpha^2 \equiv 1 \bmod m$$

is 1 mod m.

This completes the proof of Theorem 1.

The complexity in Theorem 2 is due to the fact that, when n is even, not all solutions of

$$\alpha^2 \equiv 1 \bmod m$$

are n th power residues of m as may readily be seen from Lemmas 6 to 10. But we can easily show, by means of the Chinese remainder theorem, as in the proof of Theorem 1, that if a solution α of

$$\alpha^2 \equiv 1 \bmod m$$

is also an nth power residue with respect to each of $p_1^{a_1}, \dots, p_r^{a_r}$, where $p_1, p_2, \dots, p_r$ are the distinct prime factors of m (including 2), then, and only then, it will be an n th power residue of m. When p is a prime of the form $2^{\nu+1}\lambda + 1$, where ν is the greatest exponent for which 2^ν is a factor of n, we see by Lemmas 6 and 7 that both the solutions 1 and -1 of

$$\alpha^2 \equiv 1 \bmod p^a$$

are also n th power residues of p, and that when p is of

the form $2^{\mu}\lambda+1$, where λ is odd and $\mu \gtreqless \nu$, 1 is the only solution which is also an nth power residue of p^a. Further, Lemmas 8 to 10 show that 1 is the only nth power residue of 2 when $\mu \leqq \nu+2$ which satisfies

$$\alpha^2 \equiv 1 \bmod 2^{\mu},$$

and that when $\mu > \nu+2$, of the four solutions

$$1,\ 2^{\mu-1}-1,\ 2^{\mu-1}+1,\ 2^{\mu}-1$$

only the first and the third are nth power residues of 2^{μ}.

The proof of Theorem 2 is now completed exactly like that of Theorem 1, by means of Lemma 11. We shall consider just one case as deserving special mention.

Let $m = 2^{\mu}p^{a}$, where $\mu > \nu+2$ and p is a prime of the form $2^{\nu+1}\lambda+1$. Then Lemmas 7, 10 and 11 show that there are four solutions of

$$\alpha^2 \equiv 1 \bmod m$$

which are also nth power residues of m and that their product is equal to

$$(1+2^{\mu-1})^2 \equiv 1 \bmod 2^{\mu}$$

and equal to

$$(\pm 1)^2 \equiv 1 \bmod p^{a},$$

and so equal to 1 mod $2^{\mu}p^{a}$.

ON THE PROBABILITY THAT TWO kTH POWER-FREE INTEGERS BELONGING TO AN ASSIGNED ARITHMETIC PROGRESSION SHOULD BE PRIME TO ONE ANOTHER.

BY

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Hardy and Wright derived* the probability that two integers should be relatively prime from the asymptotic formula for

$$\sum_{n \leqslant x} \phi(n).$$

* *Introduction to The Theory of Numbers* (Oxford), 1938, Theorem 332.

In this paper the problem is extended to (1) kth power-free integers belonging to an assigned arithmetic progression and (2) all integers in the A.P. A kth power-free integer is defined to be an integer which is not divisible by the kth power of any integer excepting unity.

The results of this paper are :

THEOREM 1. *Let $(v, c) = 1$. The probability that two kth power-free integers chosen from the arithmetic progression $vn+c$ should be prime to one another is*

$$\prod_{\substack{p \\ (p, v)=1}} \left\{ 1-(1/p^2)(1-1/p^{k-1})^2/(1-1/p^k)^2 \right\},$$

where p runs through all the primes not dividing v.

COR. *The probability that two kth power-free integers should be prime to one another is*

$$\prod_{p} \left\{ \left[1-(1/p^2)(1-1/p^{k-1})^2\right]/(1-1/p^k)^2 \right\}.$$

THEOREM 2. *The probability that two integers chosen from the arithmetic progression $vn+c$ should be prime to one another is*

$$(6/\pi^2)/\prod_{p/v} (1-1/p^2).$$

It is obvious that the result given by Hardy and Wright referred to above is a particular case of this theorem.

For the proof of Theorem 1 we require the following

LEMMA. *Let r and d be given square-free integers. Then the number of integers m for which $0 \leqslant md \leqslant x$ and*

$$md \equiv \begin{cases} c \pmod{v} \\ 0 \pmod{r^k} \end{cases}$$

is equal to

$$\begin{cases} 0 \text{ for } (v, rd) > 1, \\ x(r, d)/r^k dv + O(1) \text{ for } (v, rd) = 1, \end{cases}$$

where the constant implied by O is independent of x, r, d and v.

PROOF. Any multiple md of d which is also a multiple of r^k can be written as

$$m_1 r^k d/(r, d)$$

since $(r^k, d) = (r, d)$, r and d being square-free.

$$m_1 r^k d/(r, d) = vn+c$$

is solvable if and only if $(v, r^k d/(r, d))/c$. This is so, if and only if $(v, rd) = 1$. Hence the first part of the lemma is evident. Let $(v, rd) = 1$ and m_0 the solution for m_1 in the interval $0 \leqslant m_0 < v$. If m_1 is any general solution,

$$(m_1 - m_0)\, r^k d/(r, d) = hv$$
$$= h_1 r^k dv/(r, d),$$

since $r^k dv/(r, d)$ is the l.c.m. of v and $r^k d/(r, d)$.

Hence any general solution m_1 is given by

$$m_0 r^k d/(r, d) + h_1 r^k dv/(r, d) = m_1 r^k d/(r, d).$$

To each m_1 corresponds a h_1 and conversely. Hence the number of values of m subject to the above restrictions is equal to the number of values of h_1 for which

$$m_0 r^k d/(r, d) + h_1 r^k dv/(r, d) \leqslant x.$$

Hence the required number $= x\,(r, d)/r^k dv + O(1)$.

PROOF OF THEOREM 1. Let $N(x, d)$ denote the number of kth power-free integers of the form $vn+c$ which are divisible by d and are not greater than x.

(i) Let $(d, v) = 1$. Then

$$N(x, d) = \sum_{\substack{m \\ md = vn+c \leqslant x}} \; \sum_{\substack{r \\ r^k h = md}} \mu(r) = \sum_{r \leqslant x^{1/k}} \mu(r) \sum_{m} 1$$

subject to

$$md \leqslant x \text{ and } md \equiv \begin{cases} c \pmod{v} \\ 0 \pmod{r^k}. \end{cases}$$

By the lemma,

$$N(x, d) = \sum_{\substack{r \leqslant x^{1/k} \\ (r, v) = 1}} \mu(r) \frac{x(r, d)}{r^k vd} + O(x^{1/k})$$
$$= \frac{x}{vd} \sum_{\delta / d} \sum_{\substack{r \leqslant x^{1/k} \\ (r, d) = \delta \\ (r, v) = 1}} \frac{\mu(r)}{r^k}(r, d) + O(x^{1/k})$$
$$= \frac{x}{vd} \sum_{\delta / d} \frac{\mu(\delta)}{\delta^{k-1}} \sum_{\substack{r_1 \leqslant x^{1/k}/\delta \\ (r_1, vd) = 1}} \frac{\mu(r_1)}{r_1^k} + O(x^{1/k}),$$

by replacing r by $r_1 \delta$. [Here $(r_1, d) = 1$ since r is square-free.]

$$= \frac{1}{\zeta(k)} \frac{x}{vd} \prod_{p/d} (1 - 1/p^{k-1}) / \prod_{p/vd} (1 - 1/p^k) + O(x^{1/k})$$

$$= \frac{1}{\zeta(k)} \frac{x}{vd} \prod_{p/v} (1 - 1/p^k)^{-1} \prod_{p/d} (1 - 1/p^{k-1})(1 - 1/p^k)^{-1} + O(x^{1/k}).$$

(ii) Also for $(d, v) > 1$, $N(x, d) = 0$.

Hence the number of pairs of kth power-free integers in the A.P. which are divisible by d and are not greater than x is given by

$$\zeta^{-2}(k) \frac{x^2}{v^2} \prod_{p/v} (1 - 1/p^k)^{-2} \frac{1}{d^2} \prod_{p/d} (1 - 1/p^{k-1})^2 (1 - 1/p^k)^{-2}$$
$$+ O\left(\frac{x^{1+1/k}}{d}\right) \text{ for } (d, v) = 1$$

and 0 for $(d, v) > 1$.

Hence the number of pairs of kth power-free integers of the A.P. which are mutually prime and not greater than x is given by

$$F(x) = \zeta^{-2}(k) \frac{x^2}{v^2} \prod_{p/v} (1 - 1/p^k)^{-2}$$
$$\times \sum_{\substack{d \leq x \\ (d, v) = 1}} \frac{\mu(d)}{d^2} \prod_{p/d} (1 - 1/p^{k-1})^2 (1 - 1/p^k)^{-2}$$
$$+ O(x^{1+1/k} \log x)$$

$$= \zeta^{-2}(k) \frac{x^2}{v^2} \prod_{p/v} (1 - 1/p^k)^{-2}$$
$$\times \prod_{\substack{p \\ (p, v) = 1}} \left\{ 1 - (1/p^2) (1 - 1/p^{k-1})^2 (1 - 1/p^k)^{-2} \right\}$$
$$+ O(x^{1+1/k} \log x).$$

Putting $d = 1$ in (i) we get the number of kth power-free integers in the A.P. that are not greater than x to be equal to

$$\zeta^{-1}(k) \frac{x}{v} \prod_{p/v} (1 - 1/p^k)^{-1} + O(x^{1/k}).$$

Hence the number of pairs of kth power-free integers in the A.P. which are not greater than x is given by

$$T(x) = \zeta^{-2}(k) \frac{x^2}{v^2} \prod_{p/v} (1-1/p^k)^{-2} + O(x^{1+1/k}).$$

Hence the probability that a pair of kth power-free integers of the form $vn+c$ should be relatively prime is

$$\operatorname*{Lt}_{x\to\infty} \frac{F(x)}{T(x)} = \prod_{\substack{p\\(p,\,v)=1}} \left\{1-p^{-2}(1-1/p^{k-1})^2(1-1/p^k)^{-2}\right\}.$$

This proves Theorem 1.

PROOF OF THEOREM 2.

The number of integers in the A.P. less than x and divisible by a d, with $(v, d) = 1$ is $x/vd+O(1)$.

Hence the number of pairs of integers in the A.P. each less than x and divisible by d, with $(v, d) = 1$ is

$$x^2/v^2d^2+O(x/vd).$$

Hence the number of pairs of integers in the A.P. each less than x and prime to one another is given by

$$F(x) = \frac{x^2}{v^2} \sum_{\substack{d \leqslant x\\(d,\,v) = 1}} \frac{\mu(d)}{d^2} + O\left(\frac{x}{v}\log x\right)$$

$$= \frac{6}{\pi^2}\frac{x^2}{v^2} \prod_{p/v} (1-1/p^2)^{-1} + O(x\log x).$$

Also the total number of pairs of integers in the A.P., each of which is less than x is given by

$$T(x) = x^2/v^2+O(x/v).$$

Hence the probability that two integers chosen from the A.P. shall be relatively prime is

$$\operatorname*{Lt}_{x\to\infty} \frac{F(x)}{T(x)} = \frac{6}{\pi^2} \prod_{p/v} (1-1/p^2)^{-1}.$$

TRANSFORMATION OF PRODUCTS OF ϑ-FUNCTIONS.

BY

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1. We shall adopt the following well-known* notations for ϑ-functions:

$$q = e^{\pi i\tau}, \ \mathrm{Im}\,\tau > 0;$$
$$\vartheta_2(x|\tau) = \sum_n q^{(n+\frac{1}{2})^2} e^{(2n+1)x\pi i},$$
$$\vartheta_3(x|\tau) = \sum_n q^{n^2} e^{2nx\pi i},$$
$$\vartheta_4(x|\tau) = \sum_n (-1)^n q^{n^2} e^{2nx\pi i}.$$

Here, as well as in the sequel, $\sum_n$ denotes summation over all integers from $-\infty$ to $+\infty$.

It has been proved by Schröter† that α, β being positive integers

$$\vartheta_3(x|\alpha\tau)\,\vartheta_3(y|\beta\tau)$$
$$= \sum_{r=0}^{\alpha+\beta-1} q^{\beta r^2} e^{2r i\pi y}\,\vartheta_3[x+y+\alpha\beta r\tau|(\alpha+\beta)\tau]$$
$$\times\vartheta_3[\alpha y-\beta x+r\alpha\beta\tau|\alpha\beta(\alpha+\beta)\tau]. \quad (1)$$

In § 3 a generalization of the above is obtained in the form

$$\vartheta_3(x|\alpha\tau)\,\vartheta_3(y|\beta\tau)$$
$$= \sum_{r=0}^{M-1} q^{r^2(\alpha b^2+\beta d^2)} e^{2r\pi i(bx+dy)}$$
$$\times\vartheta_3(ax+cy+rN\tau|M\tau)$$
$$\times\vartheta_3(a\alpha y-c\beta x+r\alpha\beta\tau|\alpha\beta M\tau), \quad (2)$$

where α, β are positive integers, a, b, c, d are any integers satisfying the relation $ad-bc = \pm 1$,

* Tannery et Molk: *Function Elliptiques.*

† ,, ,, ,, Tome II, P. 166.

and $$M = a^2\alpha + c^2\beta,\ N = ab\alpha + cd\beta.$$

Schroter's result is the particular case for $a = c = d = 1$, $b = 0$.

The principle in the proof of (2) is explained in §2. In §4 some special cases of (2) are considered in detail. Most of these and the results obtained from them do not seem to be derivable, at any rate directly, from Schröter's result. The rest of the paper is concerned with further generalizations of (2).

2. Let

$$\sum_{m_1} \ldots \sum_{m_k} f(m_1, \ldots, m_k)$$

be an absolutely convergent multiple series, and

$$\begin{vmatrix} a_{11} & \ldots & a_{1k} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{k1} & \ldots & a_{kk} \end{vmatrix}$$

an integer matrix whose determinant is ± 1. Then the equations

$$m_i = a_{i1}\mu_1 + \ldots + a_{ik}\mu_k \quad (i = 1, 2, \ldots k)$$

determine $\mu_1, \ldots, \mu_k$ uniquely as integers which run independently through the set of all integers along with $m_1, \ldots, m_k$. It follows that if we write

$$f(m_1, \ldots, m_k) = f(\Sigma a_{1j}\mu_j, \ldots, \Sigma a_{kj}\mu_j) = \phi(\mu_1, \ldots, \mu_k),$$

then $$\sum_{m_1} \ldots \sum_{m_k} f(m_1, \ldots, m_k) = \sum_{\mu_1} \ldots \sum_{\mu_k} \phi(\mu_1, \ldots, \mu_k),$$

since the left hand side is absolutely convergent.

This special regrouping of terms has a remarkable effect when applied to the reduction of the products of ϑ-functions.

3. Let us consider the product

$$\vartheta_3(x|\alpha\tau)\,\vartheta_3(y|\beta\tau) = \sum_m \sum_n q^{\alpha m^2 + \beta n^2} e^{2\pi i (mx+ny)}, \tag{3}$$

where α, β are positive integers. Let further a, b, c, d be four integers such that

$$ad-bc=\pm 1. \tag{4}$$

Then, as explained in § 2, the second member of (3) is equal to

$$\sum_{\mu}\sum_{\nu} q^{\alpha(a\mu+b\nu)^2+\beta(c\mu+d\nu)^2}\, e^{2\pi i\,[x\,(a\mu+b\nu)+y\,(c\mu+d\nu)]} \tag{5}$$

Using (4) it is readily seen that

$$\alpha(a\mu+b\nu)+\beta(c\mu+d\nu)^2=M\,(\mu+N\nu/M)^2+\alpha\beta\nu^2/M \tag{6}$$

and $$x(a\mu+b\nu)+y\,(c\mu+d\nu)=\xi(\mu+N\nu/M)+\nu\eta/M, \tag{7}$$

where

$$M=a^2\alpha+c^2\beta,\ N=ab\alpha+cd\beta,\ \xi=ax+cy,\ \eta=a\alpha\,y-c\beta x. \tag{8}$$

Hence (5) may be written as

$$\sum_{\nu}\Big[q^{\alpha\beta\nu^2/M}\ e^{2\pi i\nu\eta/M}\ \sum_{\mu} q^{M(\mu+N\nu/M)^2}\ e^{2\pi i\xi\,(\mu+N\nu/M)}\Big]. \tag{9}$$

In the inner summation in (9) μ may clearly be replaced by $\mu+l$, l being any integer, without affecting the sum.

Therefore, writing

$$\nu=\lambda M+r,\ 0\leqq r < M,$$

and summing separately over all ν's belonging to the same residue class mod M, we see that the expression (9) is equal to

$$\sum_{r=0}^{M-1}\sum_{\lambda}\Big[q^{M\alpha\beta(\lambda+r/M)^2}e^{2\pi i\eta\,(\lambda+r/M)}\sum_{\mu} q^{M(\mu+Nr/M)^2}e^{2\pi i\xi\,(\mu+Nr/M)}\Big].$$

Thus we get

$$\vartheta_3(x|\alpha\tau)\ \vartheta_3(y|\beta\tau)$$

$$=\sum_{r=0}^{M-1}\Big[\sum_{\mu} q^{M(\mu+Nr/M)^2}e^{2\pi\xi\,(\eta+Nr/M)}\Big]$$

$$\times\Big[\sum_{\lambda} q^{M\alpha\beta\,(\lambda+r/M)^2}e^{2\pi i\eta\,(\lambda+r/M)}\Big]. \tag{10}$$

The proof of (2) is complete if we observe that

$$\Big[\sum_{\mu} q^{M(\mu+Nr/M)^2}e^{2\pi i\xi(\mu+Nr/M)}\Big]\Big[\sum_{\lambda} q^{M\alpha\beta\,(\lambda+r/M)^2}e^{2\pi i\eta(\lambda+r/M)}\Big]$$

$$= q^{r^2(N^2+\alpha\beta)/M} e^{2\pi i r\,(N\xi+\eta)/M} \vartheta_3(N\tau r+\xi|M\tau)$$

$$\times\ \vartheta_3(\alpha\beta r\tau+\eta|M\alpha\beta\tau),$$

and that $N^2+\alpha\beta = M(b^2\alpha+d^2\beta)$, $N\xi+\eta = M\ (bx+dy)$. (11)

.4. Solving for x and y in terms of ξ and η from (8) we get

$$x = \frac{a\alpha\xi - c\eta}{M},\ y = \frac{a\eta+c\beta\xi}{M}$$

Hence (10) may be written as

$$\vartheta_3\left[\frac{\alpha a\xi - c\eta}{M}|\alpha\tau\right]\vartheta_3\left[\frac{a\eta+c\beta\xi}{M}|\beta\tau\right] = \sum_{r=1}^{M-1} \phi_{Nr}(\xi)\psi_r(\eta), \qquad (12)$$

where

$$\left.\begin{aligned} \psi_r(\eta) &= \sum_\mu q^{M\alpha\beta\,(\mu+r/M)^2} e^{2\pi i\eta\,(\mu+r/M)} \\ \text{and}\quad \phi_r(\xi) &= \sum_\mu q^{M(\mu+r/M)^2} e^{2\pi i\xi\,(\mu+r/M)}. \end{aligned}\right\}(13)$$

It is readily seen that

$$\left.\begin{aligned} \psi_{M-r}(\eta) &= \psi_r(-\eta) \\ \text{and}\quad \phi_{n(m-r)}(\xi) &= \phi_{nr}(-\xi). \end{aligned}\right\}. \qquad (14)$$

We note also that

$$\phi_{nr}(\xi) = \phi_\sigma(\xi), \qquad (15)$$

where σ is any number congruent to Nr mod M.

Hence (12) may be written as

$$\vartheta_3\left[\frac{a\alpha\xi - c\eta}{M}|\alpha\tau\right]\vartheta_3\left[\frac{a\eta+c\beta\xi}{M}|\beta\tau\right]$$

$$= \begin{cases} \phi_0(\xi)\psi_0(\eta) + \displaystyle\sum_{r=1}^{(M-1)/2} \Big[\phi_\sigma(\xi)\,\psi_r(\eta)+\phi_\sigma(-\xi)\,\psi_r(-\eta)\Big], \\ \qquad\qquad\qquad\qquad \text{if } M \text{ is odd} \\ \phi_0(\xi)\psi_0(\eta) + \phi_{NM/2}(\xi)\,\psi_{M/2}(\eta) \\ \qquad + \displaystyle\sum_{r=1}^{M/2-1} \phi_\sigma(\xi)\psi_r(\eta)+\phi_\sigma(-\xi)\psi_r(-\eta)\Big], \\ \qquad\qquad\qquad\qquad \text{if } M \text{ is even,} \end{cases} \qquad (16)$$

where

$$\sigma \equiv N r \bmod M. \tag{17}$$

Let us now suppose that M is an odd prime and put $\xi = 0$ in (16). We get

$$\vartheta_3\left(\frac{c\eta}{M}\middle|\alpha\tau\right)\vartheta_3\left(\frac{a\eta}{M}\middle|\beta\tau\right)$$

$$= \phi_0(0)\,\psi_0(\eta) + \sum_{r=1}^{(M-1)/2} \phi_\sigma(0)\left[\psi_r(\eta)+\psi_r(-\eta)\right]. \tag{18}$$

We further write

$$N r \equiv \sigma \equiv \pm\rho \bmod M,\ 0 \leqslant \rho \leqslant (M-1)/2 \tag{19}$$

and notice that

$$\phi_\rho(0) = \phi_{-\rho}(0). \tag{20}$$

Also, from (11) we get

$$N^2 + \alpha\beta \equiv 0 \bmod M.$$

Multiplying by r^2 this becomes

$$\rho^2 + \alpha\beta r^2 \equiv 0 \bmod M. \tag{21}$$

$M\,[= a^2\alpha + c^2\beta]$ being prime, it is in general also prime to $\alpha\beta$ and N. The exceptional case occurs when α or β, say the former, equals M so that $a = \mp 1$, $c = 0$, $N \equiv 0 \bmod M$ and (18) reduces to a trivial identity. We shall therefore exclude this exceptional case once for all by the assumption that α and β are both less than M. Then if the prime M is equal to $a^2\alpha + c^2\beta$, it is clear that neither a nor c can be zero. Further, as r runs through the numbers 1 to $(M-1)/2$, ρ also does so since N is prime to M. Moreover it is easily seen from the congruence relation (21) that ρ and r, being integers not greater than $(M-1)/2$, determine each other uniquely. Thus from relations (18) to (21) we get the following

THEOREM I. *If M is an odd prime which can be expressed in the form $a^2\alpha + c^2\beta$, α and β being positive integers, then*

$$\vartheta_3\left(\frac{c\eta}{M}\middle|\alpha\tau\right)\vartheta_3\left(\frac{a\eta}{M}\middle|\beta\nu\right)$$

$$= \phi_0(0)\,\psi_0(\eta) + \sum_{r=1}^{(M-1)/2} \phi_\rho(0)\,[\psi_r(\eta)+\psi_r(-\eta)], \qquad (22)$$

where ρ is the unique positive solution, less than or equal to $(M-1)/2$ of (21), and ϕ_r and ψ_r are given by (13).

As the second member in (22) does not involve a and c directly but only in the combination $M=a^2\alpha+c^2\beta$, it follows that if $a'^2\alpha+c'^2\beta$ is another possible representation of the prime M, then

$$\vartheta_3\left(\frac{c\eta}{M}\middle|\alpha\tau\right)\vartheta_3\left(\frac{a\eta}{M}\middle|\beta\tau\right)=\vartheta_3\left(\frac{c'\eta}{M}\middle|\alpha\tau\right)\vartheta_3\left(\frac{a'\eta}{M}\middle|\beta\tau\right)$$

for all values of η. But this implies that

$$a=\pm a',\ c=\pm c' \qquad (23)$$

if α and β are unequal, and

$$\left.\begin{array}{l} a=\pm a',\ c=\pm c' \\ \text{or} \\ a=\pm c',\ c=\pm a' \end{array}\right\} \qquad (24)$$

if α, β are equal and therefore each equal to 1, since any common factor of α, β must also be a factor of M. Hence we get the following

THEOREM 2.* *If an odd prime can be represented in the form $a^2\alpha+c^2\beta$, where α and β are given positive integers, then this representation is unique apart from its trivial variations given by* (23) *or* (24).

The theorem, of course, remains true even if the prime is equal to α or β.

Taking $\eta=0$ in (22) we get

THEOREM 3. *If M is an odd prime which can be expressed in the form $a^2\alpha+c^2\beta$, where α, β are given positive integers, then*

$$\vartheta_3(0|\alpha\tau)\,\vartheta_3(0|\beta\tau)-\vartheta_3(0|M\tau)\,\vartheta_3(0|M\alpha\beta\tau)$$
$$=2\sum_{r=1}^{(M-1)/2} q^{(\rho^2+\alpha\beta r^2)/M}\vartheta_3(\rho\tau|M\tau)\,\vartheta_3(\alpha\beta r\tau|\alpha\beta M\tau),$$

* This is a known result in the Theory of Binary quadratic form.

where ρ is the unique positive solution, not greater than $(M-1)/2$ of

$$\rho^2+\alpha\beta r^2 \equiv 0 \bmod M.$$

From Theorem 3, by equating coefficients we at once deduce

THEOREM 4. *If M is an odd prime of the form $a^2\alpha+c^2\beta$ where α, β are given positive integers and $D(N)$ and $D(N;\, r, \rho)$ denote respectively the number of representations of N in the forms*

$$\alpha m^2+\beta n^2$$

and

$$M\left\{\left(m+\frac{\rho}{M}\right)^2+\alpha\beta\left(n+\frac{r}{M}\right)^2\right\},$$

where m, n are positive or negative integers or zero and ρ is defined as in Theorem 3, then

$$D(N) = D(N;\, 0, 0)+2\sum_{r=1}^{(M-1)/2} D(N;\, r, \rho).$$

Let us take $\eta = M\alpha\beta\tau/2$ in (22). It is obvious that if an odd prime is of the form $a^2\alpha+c^2\beta$, then one of $a\alpha$, $c\beta$ is odd and the other is even. Hence, after some reduction making use of known relations between the ϑ-functions, we get

THEOREM 5. *If M is an odd prime of the form $a^2\alpha+c^2\beta$ where α, β are given positive integers, then*

$$\vartheta_3(0/M\tau)\,\vartheta_3(0/M\alpha\beta\tau) + \sum_{r=1}^{(M-1)/2} q^{(\rho^2+\alpha\beta r^2)/M}\vartheta_3(\rho\tau/M\tau)\,\vartheta_2(r\alpha\beta\tau/M\alpha\beta\tau)$$

$$=\begin{cases}\vartheta_3(0/\alpha\tau)\vartheta_2(0/\beta\tau) \text{ if } a\alpha \text{ is odd and } c\beta \text{ is even,}\\ \vartheta_2(0/\alpha\tau)\vartheta_3(0/\beta\tau) \text{ if } a\alpha \text{ is even and } c\beta \text{ is odd,}\end{cases}$$

where ρ has the same meaning as in Theorem 3.

Again, let us take $\mu = M/2$ in (22). M being odd both a and c cannot be even; and a little reduction gives

THEOREM 6. *M and ρ being the same as in Theorems 3, 4 and 5 we have*

$$\vartheta_3(0/M\tau)\vartheta_4(0/M\alpha\beta\tau) + \sum_{r=1}^{(M-1)/2} (-1)^r/q^{(\rho^2+\alpha\beta r^2)/M}\vartheta_3(\rho\tau|M\tau)\vartheta_4(r\alpha\beta\tau|M\alpha\beta\tau)$$

$$= \begin{cases} \vartheta_4(0 \mid \alpha\tau)\vartheta_4(0 \mid \beta\tau) & \text{if both } a \text{ and } c \text{ are odd,} \\ \vartheta_4(0 \mid \alpha\tau)\vartheta_3(0 \mid \beta\tau) & \text{if } a \text{ is even and } c \text{ is odd,} \\ \vartheta_3(0 \mid \alpha\tau)\vartheta_4(0 \mid \beta\tau) & \text{if } a \text{ is odd and } c \text{ is even.} \end{cases}$$

Let us take $\alpha = 1$, $\beta = M-1$. Then the relation $M = a^2\alpha + c^2\beta$ gives $a = \pm 1$, $c = \pm 1$.

Also the congruence relation (21) becomes

$$\rho^2 - r^2 \equiv 0 \bmod M$$

so that $\rho = r \cdot (r = 1, 2, \ldots. (M-1)/2)$. Hence Theorems 3 and 6 reduce to

THEOREM 7. *If M is an odd prime, then*

$$\vartheta_3(0 \mid \tau)\,\vartheta_3(0 \mid (M-1)\tau) - \vartheta_3(0 \mid M\tau)\vartheta_3(0 \mid M(M-1)\tau)$$
$$= 2\sum_{r=1}^{(M-1)/2} q^{r^2}\vartheta_3(r\tau \mid M\tau)\vartheta_3(r(M-1)\tau \mid M(M-1)\tau)\,;$$

$$\vartheta_3(0|\tau)\,\vartheta_2(0 \mid (M-1)\tau) - \vartheta_3(0 \mid M\tau)\vartheta_2(0 \mid M(M-1)\tau)$$
$$= 2\sum_{r=1}^{(M-1)/2} q^{r^2}\vartheta_3(r\tau \mid M\tau)\vartheta_2(r(M-1)\tau \mid M(M-1)\tau)\,;$$

$$\vartheta_4(0|\tau)\,\vartheta_4(0 \mid (M-1)\nu) - \vartheta_3(0 \mid M\tau)\vartheta_4(0 \mid M(M-1)\tau)$$
$$= 2\sum_{r=1}^{(M-1)/2} (-1)^r q^{r^2}\vartheta_3(r\tau| M\tau)\vartheta_4(r(M-1)\tau|M(M-1)\tau).$$

Similarly Theorem 4 becomes

THEOREM 8. *If M is an odd prime and $R(N)$, $R(N, r)$ denote respectively the number of representations of N in the forms*

$$m^2 + (M-1)n^2$$

and

$$M[(m+r/M)^2 + (M-1)\,(n+r/M)^2],$$

where m and n are positive or negative integers or zero, then

$$R(N) = R(N, 0) + 2 \sum_{r=1}^{(M-1)/2} R(N, r)$$

Another special case of some interest is obtained on taking $\alpha = \beta = 1$. The congruence relation (21) then reduces to

$$\rho^2 + r^2 \equiv 0 \bmod M. \tag{25}$$

It is known that the necessary and sufficient condition for representing an odd prime as the sum of two integral squares is that the prime should be of the form $4p+1$. Also, the congruence relation (25) shows that for $1 \leqslant r, \rho \leqslant (M-1)/2$, ρ cannot be equal to r, since otherwise $2r^2 \equiv \bmod M$ so that $r \equiv 0 \bmod M$ which is impossible. Hence we get from Theorem 3 the following

THEOREM 9. *If M is a prime of the form $4p+1$, then*

$$\vartheta_3^2(0 \mid \tau) - \vartheta_3^2(0 \mid M\tau) = 4 \sum_{(r,\rho)} q^{(r^2+\rho^2)/M} \vartheta_3(r\tau \mid M\tau)\, \vartheta_3(\rho\tau \mid M\tau),$$

where $\sum_{(r,\rho)}$ denotes summation over all the $(M-1)/4$ distinct pairs of positive numbers r, ρ, not greater than $(M-1)/2$ and satisfying the relation

$$\rho^2 + r^2 \equiv 0 \bmod M. \tag{25}$$

Thus, for instance, taking $M = 5$, we get

$$\vartheta_3^2(0 \mid \tau) - \vartheta_3^2(0 \mid 5\tau) = 4q\vartheta_3(\tau \mid 5\tau)\vartheta_3(2\tau \mid 5\tau).$$

From Theorem 9, by equating coefficients we at once deduce

THEOREM 10. *If M is a prime of the form $4p+1$ and $T(N)$ and $T(N; r, \rho)$ denote respectively the number of representations of N in the forms*

$$m^2 + n^2$$

and

$$M[(m + r/M)^2 + (n + \rho/M)^2],$$

where m and n are positive or negative integers or zero, and r, ρ satisfy (25), then

$$T(N) = T(N; 0, 0) + 4 \sum_{1(r,\rho)} T(N; r, \rho),$$

$\sum\limits_{(r,\rho)}$ having the same meaning as in Theorem 9.

In a similar manner, when $\alpha=\beta=1$, Theorems 5 and 6 reduce to the following

THEOREM 11: *If M is a prime of the form $4p+1$, then*

$$\vartheta_3(0\,|\,\tau)\,\vartheta_2(0\,|\,\tau)-\vartheta_3(0\,|\,M\tau)\,\vartheta_2(0\,|\,M\tau)$$
$$=2\sum_{r=1}^{(M-1)/2} q^{(r^2+\rho^2)/M}\vartheta_3(\rho\tau\,|\,M\tau)\,\vartheta_2(r\tau\,|\,M\tau)\ ;$$
$$\vartheta_3(0\,|\,\tau)\,\vartheta_4(0\,|\,\tau)-\vartheta_3(0\,|\,M\tau)\,\vartheta_4(0\,|\,M\tau)$$
$$=2\sum_{r=1}^{(M-1)/2}(-1)^r q^{(r^2+\rho^2)/M}\vartheta_3(\rho\tau\,|\,M\tau)\,\vartheta_4(r\tau\,|\,M\tau),$$

where ρ is the unique positive solution, not greater than $(M-1)/2$ of

$$\rho^2+r^2\equiv 0 \bmod M. \tag{25}$$

5. We shall now obtain a further generalization of (2). Let $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ be positive integers $\|a_{ij}\|$ an integer matrix of order n whose determinant is ± 1. Now consider the product

$$\prod_{j=1}^{n}\vartheta_3(x_j\,|\,\alpha_j\tau)=\sum_{m_1}\cdots\sum_{m_n} q^{\Sigma\alpha_j m_j^2}\, e^{2\pi i\Sigma m_j x_j}. \tag{26}$$

By what has been said in § 2, the second member of (26) is equal to

$$\sum_{\mu_1}\cdots\sum_{\mu_n} q^{\Sigma\alpha_j(a_{j1}\mu_1+\ldots+a_{jn}\mu_n)^2}\, e^{2\pi i\,\Sigma x_j(a_{j1}\mu_1+\ldots+a_{jn}\mu_n)}. \tag{27}$$

Let us write

$$P=\sum_{j=1}^{n}\alpha_j\,(a_{j1}\mu_1+\ldots+a_{jn}\mu_n)^2$$
$$=\sum_{j=1}^{n}\sum_{k=1}^{n}A_{jk}\mu_j\mu_k,\quad A_{jk}=A_{kj},$$

and

$$Q=\sum_{j=1}^{n}x_j\,(a_{j1}\mu_1+\ldots+a_{jn}\mu_n)^2$$

$$= \sum_{j=1} \xi_j \mu_j.$$

Then by a well-known theorem* on the reduction of quadratic forms, we can write

$$P = \sum_{j=1}^{n} z_j^2/(D_{j-1}D_j),$$

where $D_0 = 1$, D_j is the determinant

$$\begin{vmatrix} A_{11} \ldots A_{1j} \\ \vdots \\ A_{j1} \ldots A_{jj} \end{vmatrix}$$

and z_j is the determinant obtained by replacing the elements in the last row or column of D_j by $X_1, X_2, \ldots, X_j$ respectively, X_i standing for the expression

$$A_{i1}\mu_1 + \ldots + A_{in}\mu_n.$$

It is easy to see that z_j is independent of $\mu_1, \ldots, \mu_{j-1}$, that the coefficient of μ_j in it is equal to D_j and that the coefficient of μ_{j+k} $(k = 1, 2, \ldots, n-j)$ in it is equal to the determinant

$$D_{jk} = \begin{vmatrix} A_{11} \ldots A_{1j-1} & A_{1j+k} \\ \vdots & \\ A_{j1} \ldots A_{j,j-1} & A_{j,j+k} \end{vmatrix}$$

We may therefore write P in the form

$$D_1\left[\mu_1 + \frac{D_{12}}{D_1}\mu_2 + \ldots + \frac{D_{1n}}{D_1}\mu_n\right]^2$$

$$+ D_1 D_2\left[\frac{\mu_2}{D_1} + \frac{D_{23}}{D_1 D_2}\mu_3 + \ldots + \frac{D_{2n}}{D_1 D_2}\mu_n\right]^2$$

$$+ D_1^2 D_2 D_3\left[\frac{\mu_3}{D_1 D_2} + \frac{D_{34}}{D_1 D_2 D_3}\mu_4 + \ldots + \frac{D_{3n}}{D_1 D_2 D_3}\mu_n\right]^2$$

$$+ \ldots\ldots$$

$$+ D_1^2 \ldots D_{n-2}^2 D_{n-1} D_n\left[\frac{\mu_n}{D_1 D_2 \ldots D_{n-1}}\right]^2.$$

Denoting $D_0 D_1 \ldots D_j$ by ∂_j $(j = 0, 1, \ldots, n)$ we may finally write

* *Vide* Bromwich : *Quadratic forms and their classifications*, p. 16.

$$P = \partial_1 \left[\mu_1 + \frac{D_{12}}{\partial_1}\mu_2 + \ldots + \frac{D_{1n}}{\partial_1}\mu_n \right]^2 .$$

$$+ \sum_{j=1}^{n-1} \partial_{j-1}\partial_{j+1} \left[\frac{\mu_{j+1}}{\partial} + \frac{D_{j+1,\,j+2}}{\partial_{j+1}}\mu_{j+2} + \ldots + \frac{D_{j+1,\,n}}{\partial_{j+1}}\mu_n \right]^2 . \quad (28)$$

Let us also write Q in the form

$$\sum_{j=1}^{n} \eta_j \left\{ \frac{\mu_j}{\partial_{j-1}} + \frac{D_{j,\,j+1}}{\partial_j}\mu_{j+1} + \ldots + \frac{D_{jn,\,n}}{\partial_j}\mu_n \right\}, \quad (29)$$

where the η's are linear homogeneous functions of the ξ's and hence also of the x's.

We now write

$$\mu_1 = \lambda_1$$
$$\mu_i = \lambda_i \delta_{i-1} + \nu_{i-1}$$
$$0 \leqslant \nu_{i-1} < \delta_{i-1} \ (i = 2, 3, \ldots, n),$$

and evaluate (27) by carrying out the summations for $\mu_1, \mu_2, \ldots, \mu_n$ successively, that for each μ_i $(i = 1, \ldots, n)$ being performed by taking the sum over all numbers belonging to the same residue class, mod δ_{i-1}, separately. In this way we see, after a little reduction, that (27) is equal to

$$\sum_{\nu_1=0}^{\delta_1-1} \cdots \sum_{\nu_{n-1}=0}^{\delta_{n-1}-1} \prod_{j=1}^{\mu} \sum_{\lambda_j} q^{P_j} e^{2\pi i Q_j}, \quad (30)$$

where

$$P_1 = \delta_1 [\lambda_1 + \frac{D_{12}}{\delta_1}\nu_1 + \ldots + \frac{D_{1n}}{\delta_1}\nu_{n-1}],$$

$$P_j = \delta_{j-2}\delta_j [\lambda_j + \frac{\nu_{j-1}}{\delta_{j-1}} + \frac{D_{j,\,j+1}}{\delta_j}\nu_j + \ldots + \frac{D_{j,\,n}}{\delta_j}\nu_{n-1}]^2,$$

$$(j = 2, 3, \ldots, n)$$

and

$$Q_1 = \eta_1 [\lambda_1 + \frac{D_{12}}{\delta_1}\nu_1 + \ldots + \frac{D_{1n}}{\delta_1}\nu_n]$$

$$Q_j = \eta_j [\lambda_j + \frac{\nu_{j-1}}{\delta_{j-1}} + \frac{D_{j,\,j+1}}{\delta_j}\nu_j + \ldots + \frac{D_{j,\,n}}{\delta_j}\nu_{n-1}] \ (j = 2, 3, \ldots, n).$$

The expression (30) for the value of

$$\prod_{j=1} \vartheta_3 (x_j | \alpha_j \tau)$$

is the generalization sought for.

We may further remark that the above analysis can be applied to any expression of the form

$$\sum_{\mu_1} \cdots \sum_{\mu_n} q^P e^{2\pi i Q},$$

where P is a positive definite quadratic form in $\mu_1, \ldots, \mu_n$ with integral coefficients and Q is linear in $\mu_1, \ldots, \mu_n$. It follows in particular that the above expression can be expressed as a sum of products of ϑ-series with elementary functions as coefficients

ON A CERTAIN SCROLL ASSOCIATED WITH A NET OF QUADRICS.

BY

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1. In a paper published in the *Bulletin of the Calcutta Mathematical Society* (Chariar, 1944) the author discussed a scroll generated by lines whose polars with respect to a net of quadrics are concurrent. It was proved that the surface R generated by such lines is of order 9, having the Jacobian curve Γ of the net as a multiple curve. In this paper some other properties of R are obtained. It is found that R is generated by trisectants of Γ which is the only multiple curve on R. It is also found that Γ is a multiple curve of order three on R whose genus is 10. Lines whose polars with respect to four quadrics not belonging to a net, are concurrent are also discussed.

2. If the polar lines of a line l with respect to a net of quadrics are concurrent at P, the polar planes of P with respect to the quadrics of the net have a common line of section l and therefore P lies on the Jacobian Γ of the net and conversely the polar planes of P, a point on

Γ, have a common line of section $l(P)$ whose polars with respect to the members of the net of quadrics are concurrent at P. *Hence R is generated by lines corresponding to points on Γ—the Jacobian of the net of quadrics.* Taking P as the origin of cartesian coordinates so that one of the quadrics is a cone with P as vertex and the line $l(P)$ as given by $y=\beta$, $z=0$ we have the equations of the three quadrics forming the net given by

$$S_1 \equiv a_1x^2+b_1y^2+c_1z^2=0,$$
$$S_2 \equiv (a_2, b_2, c_2, f_2, g_2, h_2, 0, v_2, w_2, -v_2\beta)(x, y, z, 1)^2=0,$$
$$S_3 \equiv (a_3, b_3, c_3, f_3, g_3, h_3, 0, v_3, w_3, -v_3\beta)(x, y, z, 1)^2=0.$$

The Jacobian of the net is given by

$$\left\| \begin{matrix} a_1x & h_1y & c_1z & 0 \\ (a_2x) & (h_2x) & (g_2x) & (u_2x) \\ (a_3x) & (h_3x) & (g_3x) & (u_3x) \end{matrix} \right\| = 0,$$

where $(a_2x) \equiv a_2x+h_2y+g_2z+u_2$, etc.

Three of the four cubics which intersect in the Jacobian of the net, have the line $l(P)$ as a common generator and $l(P)$ meets Γ the common curve of intersection of the four cubics in three points given by

$$\begin{vmatrix} a_1x & b_1\beta & 0 \\ a_2x+h_2\beta & h_2x+b_2\beta+v_2 & g_2x+f_2\beta+w_2 \\ a_3x+h_3\beta & h_3x+b_3\beta+v_3 & g_3x+f_3\beta+w_3 \end{vmatrix} = 0.$$

Hence R is generated by certain trisectants of the Jacobian of the net of quadrics which are the common lines of section of polar planes of points of the Jacobian.

3. Let now ABC be a trisectant of Γ, the points A, B, C being distinct. The line $l(A)$ corresponding to A, is the line of intersection of the polar planes of A with regard to any two members of the net and therefore is the line of intersection of the polar planes of the line ABC with respect to the two cones $c(B)$, $c(C)$ (whose vertices are B and C) belonging to the net. And similarly for $l(B)$ and $l(C)$. Hence the lines $l(A)$, $l(B)$, $l(C)$ corresponding to A, B, C, are concurrent at Q which is the point of intersection of the polar planes of the line ABC with

respect to the cones $c(A)$, $c(B)$, $c(C)$. It is evident that Q lies on Γ because the polar planes of Q with regard to the net of quadrics form a pencil. *Hence R is generated by the trisectants of* Γ which meet Γ in three distinct points.

4. We have seen that if $g(P)$ be a generator of R through P a point of Γ, the polar lines of $g(P)$ with respect to the net are concurrent at a point Q of Γ to which $g(P)$ corresponds. Now because the polar planes of P must pass through Q, Q must lie on the line $l(P)$ which meets Γ in three and only three points. *Hence through any point P of* Γ, *there will pass three and only three generators of R which correspond to the three points in which the line $l(P)$ meets* Γ. *Hence* Γ *is a multiple curve of order* 3 *on R.*

5. We have noted that the generators of R are lines corresponding to points on Γ. Let two of these lines $l(P)$ and $l(Q)$ corresponding to two points P, Q on Γ meet at A so that A is a double point on R. Now the polar planes of A with respect to the quadrics of the net pass through P and Q and therefore have a common line of section PQ and hence A must lie on Γ. *Thus* Γ *is the only multiple curve on R and is of order* 3. A plane π of general position meets R in a curve of order 9 having 6 triple points where π meets Γ. *Hence R is of genus* 10. That Γ is the only multiple curve and is of order 3 can be also seen as follows. The conditions that the polar lines of a line AB with respect to the net of quadrics should be concurrent are that the six polar planes—three for A and three for B—should have a common point. The three polar planes of A either meet in a point (finite or infinite) or have a common line of section when A lies on Γ. If therefore A does not lie on Γ, we get a definite point of intersection of the three planes and consequently a unique set of values for the direction ratios of AB, provided A lies on a certain surface R of order 9. Hence Γ is the only multiple curve. Taking the coordinates of A to be (α, β, γ) the three polar planes have a common point

$(X, Y, Z, / T)$ where X, Y, Z, T are certain polynomials of degree 3 in (α, β, γ). The four cubics $X = 0$, $Y = 0$, $Z = 0$, $T = 0$ have Γ for their common curve of section. Taking three general equations of the second degree as the equations of the three fundamental quadrics defining the net, the equation of R is

$$\begin{vmatrix} (a_1 X) & (h_1 X) & (g_1 X) \\ (a_2 X) & (h_2 X) & (g_2 X) \\ (a_3 X) & (h_3 X) & (g_3 X) \end{vmatrix} = 0,$$

where $(a_1 X) = a_1 X + h_1 Y + g_1 Z + u_1 T$, etc. It is evident that every point on Γ (where $X = 0 = Y = Z = T$) is a triple point on R. Γ *is thus the only multiple curve and is of order* 3.

6. We have seen above that given three quadrics S_1, S_2, S_3, lines whose polars with respect to the net are concurrent are lines corresponding to points on the Jacobian of the net and generate a surface of order 9. If S_4 is a quadric not belonging to the net, it is, in general, not possible to draw a line whose polars with respect to the four quadrics should be all concurrent. In order that the polar line of $l(P)$ (where P is a point on the Jacobian of S_1, S_2, S_3) with respect to S_4 should pass through P, it is necessary and sufficient that the polar plane of P with respect to S_4 should contain $l(P)$. This requires that the four Jacobian curves of S_1, S_2, S_3, S_4 taken in threes should interest at P. In general, these Jacobians do not have a point in common. *Hence given four quadrics which do not form a net, no line in general can be drawn whose polars with respect to the four are concurrent. If however the four Jacobian curves have m common points, there exist m such lines which are the common lines of section of the polar planes of these m points.*

...MARA ... STAD...

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